GOOD VARIATION THEORY

A tribute to

fig.1) Alfred Tauber - fig.2) Albert E. Ingham

fig.3) Godfrey H. Hardy- fig.4) Srinivasa Ramanujan- fig.5) John E. Littlewood
Foreword

“There are only three really great English mathematicians: Hardy, Littlewood, and Hardy–Littlewood.” Harald Bohr (see [ChQ] for a proof)

The point of my theory of functions of good variation (FGV) is that I make use of the so-called little Mellin transform of a weight function. Roughly speaking, the zeros of this transform give sharp informations on the asymptotic behaviour of $\sum_{k=1}^{n} a_k$ where $a_n$ is an interesting arithmetical function whereas usually one obtains (weaker) informations by means of the analysis of singularities of the Dirichlet generating function $\sum_{n \geq 1} \frac{a_n}{n^s}$ (see for instance Perron formulas [Ten] pp 133-138).

This tauberian take on things allows me to tackle problems like RH and its generalisations while classical tauberian approaches to the PNT always hit the line $x = 1$ or the line $x = 0$ which are brick walls preventing us to venture into the critical strip with these methods. To see what is going on in the middle of the critical strip we need another viewpoint allowing us to “break free” from the gravitational attraction of the pole of the zeta function at 1. That’s the idea behind good variation theory.

From that perspective I’m indeed able to reformulate the Riemann hypothesis in precise tauberian terms linking the symmetry of the functional equation to a tauberian symmetry. This way the critical line becomes an asymptotic wise for the nontrivial zeros of little Mellin transforms of broken harmonic functions (BHF) satisfying the Hardy-Littlewood-Ramanujan criterion (HLR criterion) when the transform has a riemannian functional equation. It looks like the critical strip is a tokamak producing a magnetic field strong enough to confine the plasma (the nontrivial zeros) on the critical line.

The present note aims to summarize with details and examples my theory (there are more than 50 figures in the paper). I’m trying to separate the good from the bad among my working papers written since 2011 ([Clo1] [Clo3] [Clo4] [Clo2] [Clo5] [Clo6]). This note is subject to changes depending on comments or new ideas of presentation (the last update will be always available on my website).

As the reader may notice, the note contains many conjectures but they begin to form a coherent whole where the HLR criterion becomes a central issue. It may well be time to stop computations and to try to work out proofs for some of the important claims. I am thinking of making more rigorous the proof of theorem 2 for polynomials and of finishing the proof of the analytic conjecture for continuous functions. The conjectures of the second part of the note are still difficult but I feel that it is now a question of technique available somewhere by someone. In another hand I’m still discovering new phenomena needing to be clarified and to be put into new conjectures since splitting the problem in smaller and smaller pieces can’t be bad.

You may say: what a number of conjectures! I would reply that without experiments, analogies and iterative guesses good variation theory would never have existed in such a comprehensive way. Between experiments and the
presumed right way to do mathematics by proving conjectures each one after another I made the deliberate choice of making experiments as the backbone of the theory. So I’m rather a physicist than a mathematician here because the problem needs to do so. I have confined myself to prove simple facts in the realm of continuous functions and to ensure consistency with results from analytic number theory for aspects related to the Ingham functions.

If I had tried to prove rigourously the analytic conjecture for continuous function say three years ago I’d probably still be working on it and certainly I wouldn’t have discovered the set of precise tauberian formulas related to the Ingham functions. These formulas translate into concrete terms what RH is: a hypothesis based simultaneously on discrete and continuous mathematics. In particular I unearthed a general equivalence principle unifying the asymptotic behaviour of discrete sums and integrals in the realm of good variation theory.

How good is this theory? I don’t know to be honest. My loneliness during these years induced an absence of feedback. Specific discussions with few experts arose time to time allowing me to improve the foundations of the theory on a regular basis but are helpless to have a clear advice on the whole picture. BTW I think that the last conjectures are precise enough now and are well supported by experiments so that good variation theory should deserve more attention.

fig.6) The Ingham function \( \Phi(x) = x \lfloor x^{-1} \rfloor \) on \([0, 1]\)

As we shall see this bounded and measurable function is of paramount importance in our good variation approach to \( RH \).
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Good variation theory

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1 Introduction

Good variation theory aims to get informations on the asymptotic behaviour of the partial sum

\[ A(n) := \sum_{k=1}^{n} a_k \]

where the sequence \((a_n)_{n \geq 1}\) is given by the recurrence formula

\[ A_g(n) := \sum_{k=1}^{n} a_k g \left( \frac{k}{n} \right) = h(n) \]

where \(h\) and \(g\) are given real functions and the asymptotic behaviour of \(h\) is well known. Good variation theory belongs to the more general tauberian theory [Kor].

1.1 Primary definition of a function of good variation (FGV)

Let \(\beta \in \mathbb{R}\) and \(g\) be a bounded and measurable real function defined on \([0, 1]\) satisfying \(g(1) \neq 0\). Next define \((a_n)_{n \geq 1}\) recursively as follows

\[ A_g(n) = n^{-\beta} \]

Then we say that \(g\) is a FGV of index \(\alpha(g) \in \mathbb{R}\) if the 2 following tauberian properties hold

1. \(\beta < \alpha(g) \Rightarrow A(n) \sim C(\beta)n^{-\beta} \ (n \to \infty)\) where \(C(\beta) \neq 0\)
2. \(\beta \geq \alpha(g) \Rightarrow A(n) \ll n^{-\alpha(g)}L(n) \overset{[1]}{\sim} \)

\[ \overset{[1]}{\text{1Here } L \text{ denotes a slowly varying function i.e. } \forall x > 0 \lim_{t \to \infty} \frac{L(tx)}{L(t)} = 1. \text{ It is a regularly varying function of index 0.}}\]
When \( g \) is smooth, i.e. continuous on \([0,1]\), the problem is almost solved but has few interesting applications in number theory. It could have more applications elsewhere. What is more interesting is the non smooth case, in particular when \( g \) is related to arithmetical functions so that good variation theory allows us to tackle problems such as RH and its generalisations. Indeed considering the Ingham function \( \Phi(x) = x \left\lfloor \frac{1}{x} \right\rfloor^2 \), we have exact formulas like this one easy to prove

\[
\sum_{k=1}^{n} \frac{\mu_k}{k} \Phi \left( \frac{k}{n} \right) = n^{-1}
\]

where \( \mu \) is the Moebius function (sequence A008683 in [Slo]). From this formula good variation theory implies, using a suitable conjecture described in this paper, that we have

\[
\sum_{k=1}^{n} \frac{\mu_k}{k} \ll n^{-1/2+\varepsilon}
\]

which is an asymptotic formula equivalent to RH.

1.2 Aim of this article

The aim of this paper is twofold. In the first part it is showed that good variation theory is a consistent concept since it is proved that smooth FGV exist.

In the second part it is suggested that good variation theory is the right tool to settle the status of RH using non smooth functions.

In particular this study sheds slight on intrinsic properties of the Ingham function and not actually on properties of the zeta function although the zeta function occurs in various settings.

As we shall see good variation theory is discrete by essence but requires analytic techniques to be fully efficient and a set of deep tauberian conjectures.

It is somewhat reminiscent of the Nyman-Beurling approach to RH which is based on intrinsic properties of the fractional part function [Bal] and on the classical Mellin transform. Other ideas suggest that the Dirichlet divisor problem or the Gauss circle problem rely also upon intrinsic properties of the fractional part function [Clo2].

The following tauberian theorem of Ingham ([Kor], theorem 18.2.p.110. and for an effective version see [1_5n]) was the starting point of this study which began in 2010. Namely

\[
n_{n_0} \geq -C \land \lim_{n \to \infty} A_k(n) = \ell \Rightarrow \lim_{n \to \infty} A(n) = \ell
\]

and good variation theory consists to go under the surface of this theorem. More precisely I consider \( A_k(n) \sim n^{-\beta} \) \((n \to \infty)\) when \( \beta > 0 \) so that \( \ell = 0 \). Ingham theorem is for \( \beta = 0 \) whenever \( \ell \neq 0 \) and the PNT can be deduced from it. Many results are known when \( \beta < 0 \) using methods similar to results involving the classical convolution Mellin transform ([Kor] p. 195). When \( \beta > 0 \) however I found nothing relevant in the litterature.
Despite good variation theory looks like an usual question and many problems in tauberian theory have the same flavor, there is apparently nothing relevant in the literature directly related to the approach developped therein and allowing me to prove the more important conjectures formulated in the second part of this note. For instance nothing really useful was found regarding the nontrivial case $\beta > 0$ in [Ing2, Win, Seg2, Juk, Seg, Bin, BGT, Ing, Kor]. Although there are interesting tauberian theorems for arithmetic sums involving suitable kernels and convolutions of the classical Mellin transform for $\beta < 0$ and the limit case $\beta = 0$ (see for instance [Bin2] where sums on primes are considered), I didn’t find a clear way to apply them for my purpose and the most difficult cases. In general tauberian theorems in prime number theory require stronger conditions than those I consider here (positivity of coefficients is not a concern for me, nothing specific on monotonicity of the partial sums is required etc.) and that makes the difference.

As the reader will see my approach to good variation theory is also closely related to Euler-Cauchy linear homogeneous ordinary differential equations and difference equations analogues. This helped me a lot to extend by analogy the ideas to the non smooth cases.

So this study is also a tribute to

Leonhard Euler and Augustin Louis Cauchy

An Euler-Cauchy differential equation of degree 3

$$x^3y''' + x^2y'' + xy' + y = x^{-1}$$

whose solutions are given by

$$y(x) = Ax (\sin (\log x) + \cos (\log x)) + Bx (\sin (\log x) - \cos (\log x)) + C - x - 5x^{-1}$$

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Part I
Smooth functions

2 Introduction to part I

In section 3 it is shown that $FGV$ exist since affine functions are proved to be $FGV$ according to the primary definition and the proof is constructive (theorem 3.1). Next in section 4 it is shown that polynomials are $FGV$ (theorem 4.1) and then in section 5 it is conjectured that all continuous functions are in fact $FGV$ with the analytic conjecture for continuous functions (conjecture 5.1). In section 6 I describe the tauberian equivalence conjecture for continuous functions. In section 7 discontinuous functions with a single discontinuity are considered and are shown to be $FGV$ with the theorem 7.1.. It is a natural transition between the first part and the second part of this note.

3 Affine functions are $FGV$

In this section it is shown that $FGV$ exist since affine functions are proved to be $FGV$ with the theorem 1. It turns out that any constructive proof that the simplest functions, the affine functions, are $FGV$ is rather tedious and underlines that good variation theory is not a trivial concept.

3.1 Theorem

Let $g$ be the affine function

$$g(x) = c_1 x + c_0$$

where $c_0, c_1 > 0$ so that $\frac{c_0}{c_0 + c_1} \in ]0, 1[$. Then $g$ is a $FGV$ of index

$$\alpha(g) = \frac{c_0}{c_1 + c_0}$$

according to the primary definition of $FGV$.

3.2 Proof of theorem 3.1

We need the following lemma.

3.2.1 Lemma

Let $(x_n)_{n \geq 1}$ be a non zero sequence satisfying for $c \neq 0$

- $\prod_{k=2}^{n} x_k \sim c n^{-\alpha} \quad (n \to \infty)$
Let \((U_n)_{n \geq 1}\) be defined recursively as follows

\[
U_n = x_n U_{n-1} + y_n
\]

where we have \(y_n \sim d n^{-\beta - 1} \ (n \to \infty)\) for \(d \neq 0\). Then there are explicit constants \(k_1, k_2, k_3\) such that the following relations hold

1. \(\beta > \alpha \Rightarrow U_n \sim k_1 n^{-\alpha} \ (n \to \infty)\)
2. \(\beta = \alpha \Rightarrow U_n \sim k_2 n^{-\alpha} \log n \ (n \to \infty)\)
3. \(\beta < \alpha \Rightarrow U_n \sim k_3 n^{-\beta} \ (n \to \infty)\)

### 3.2.2 Proof of lemma 3.2.1

We can rewrite \(U_n\) explicitly

\[
U_n = y_n + \left( \prod_{k=2}^{n} x_k \right) \left( U_1 + \sum_{k=2}^{n-1} \frac{y_k}{\prod_{j=2}^{k} x_j} \right)
\]

and from the hypothesis on \(x_n, y_n\) we have

\[
\frac{y_k}{\prod_{j=2}^{k} x_j} \sim \left( \frac{d}{c} \right) k^{\alpha - \beta - 1} \ (k \to \infty)
\]

hence we have 3 cases to consider:

1. \(\beta > \alpha \Rightarrow \sum_{k=2}^{n-1} \frac{y_k}{\prod_{j=2}^{k} x_j}\) converges to \(l\) and \(U_n \sim c(l + U_1)n^{-\alpha} \ (n \to \infty)\)
2. \(\beta = \alpha \Rightarrow \sum_{k=2}^{n-1} \frac{y_k}{\prod_{j=2}^{k} x_j} \sim \frac{d}{c} \log n \ (n \to \infty)\) and \(U_n \sim d n^{-\alpha} \log n \ (n \to \infty)\)
3. \(\beta < \alpha \Rightarrow \sum_{k=2}^{n-1} \frac{y_k}{\prod_{j=2}^{k} x_j} \sim \frac{d}{c(\alpha - \beta)} n^{\alpha - \beta} \ (n \to \infty)\) and \(U_n \sim \left( \frac{d}{(\alpha - \beta)} \right) n^{-\beta} \ (n \to \infty)\)

\(\square\).

### 3.2.3 Proof of theorem 3.1.

If \(A_y(n) = f(n)\) we have

\[
c_1 \sum_{k=1}^{n} k a_k + n c_0 A(n) = n f(n)
\]

next using Abel summation

\[
\sum_{k=1}^{n} k a_k = (n + 1) A(n) - \sum_{k=1}^{n} A(k)
\]

we get

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\[ c_1(n+1)A(n) - c_1 \sum_{k=1}^{n} A(k) + nc_0A(n) = nf(n) \]

which becomes by first difference

\[ A(n) = n^{-1} \left(n - \frac{c_0}{c_0 + c_1}\right) A(n-1) + n^{-1} (nf(n) - (n-1)f(n-1)) \]

Now letting

- \( f(n) = n^{-\beta} \)
- \( x_n = n^{-1} \left(n - \frac{c_0}{c_0 + c_1}\right) \)
- \( y_n = n^{-1} (nf(n) - (n-1)f(n-1)) \)

we have

\[ \prod_{k=2}^{n} x_k = \left(\frac{c_1 + c_0}{c_1}\right) \frac{1}{n!} \prod_{k=1}^{n} \left(k - \frac{c_0}{c_0 + c_1}\right) \sim \Gamma \left(\frac{2c_1 + c_0}{c_0 + c_1}\right)^{-1} n^{-\frac{c_0}{c_0 + c_1}} \quad (n \to \infty) \]

and

\[ y_n \sim (1 - \beta)n^{-\beta-1} \quad (n \to \infty) \]

thus writing \( \alpha = \frac{c_0}{c_0 + c_1} \in ]0,1[ \) we get from the lemma for suitable constants \( k_1, k_2, k_3 \)

1. \( \beta > \alpha \Rightarrow A(n) \sim k_1 n^{-\alpha} \quad (n \to \infty) \)
2. \( \beta = \alpha \Rightarrow A(n) \sim k_2 n^{-\alpha} \log n \quad (n \to \infty) \)
3. \( \beta < \alpha \Rightarrow A(n) \sim k_3 n^{-\beta} \quad (n \to \infty) \)

consequently \( g \) is a \( \text{FGV} \) of index \( \frac{c_0}{c_0 + c_1} \) satisfying the primary definition of a function of good variation. \( \square \)

**Remark**

Note that the case \( \beta = 1 \) is very specific and if we choose \( \beta \neq 1 \) we could relax the conditions on \( c_0, c_1 \) as \( c_1 (c_1 + c_0) \neq 0 \) so that \( g \) would be a \( \text{FGV} \) of index \( \alpha(g) = \frac{c_0}{c_1 + c_0} \) which can take any real value.
3.3 Theorem

In fact we can go further and state more precise estimates for $A(n)$. By analogy and computational evidence this theorem 3.3 will allow us to state later deep conjectures related to discontinuous functions like the Ingham function (in the second part of this note). Indeed there is no fundamental difference between the little Mellin transform of a non trivial FGV which has all its non trivial zeros on a single vertical line and the little Mellin transform of an affine function which has a single real zero. The important fact here is that the value $\beta = \alpha(g) - 1$ appears to be a breakpoint working for any FGV not only affine functions.

Namely let $g(x) = c_1 x + c_0$ where $c_1, c_0 > 0$ so that $\alpha(g) = \frac{c_0}{c_1 + c_0} \in ]0, 1[$ and let $g^*(z) := \frac{\alpha}{1 - z^2} - \frac{c_0}{z}$ then if $A_g(n) = n^{-\beta}$ we have the following 7 more precise asymptotic formulas than those given in 3.2.2. including an error term and covering all possible cases:

1. $\beta < \alpha(g) - 1$
   \[ A(n) = \left( -\frac{1}{\beta g^*(\beta)} \right) n^{-\beta} + O(n^{-1-\beta}) \]

2. $\beta = \alpha(g) - 1$
   \[ A(n) = \left( -\frac{1}{\beta g^*(\beta)} \right) n^{-\beta} + O\left(n^{-\alpha(g)} \log n\right) \]

3. $\alpha(g) - 1 < \beta < 0$
   \[ A(n) = \left( -\frac{1}{\beta g^*(\beta)} \right) n^{-\beta} + O\left(n^{-\alpha(g)} \right) \]

4. $\beta = 0$
   \[ A(n) = \frac{1}{c_0} + O\left(n^{-\alpha(g)} \right) \]

5. $0 < \beta < \alpha(g)$
   \[ A(n) = \left( -\frac{1}{\beta g^*(\beta)} \right) n^{-\beta} + O\left(n^{-\alpha(g)} \right) \]

6. $\beta = \alpha(g)$
   \[ A(n) = (1 - \alpha(g)) n^{-\alpha(g)} \log n + O\left(n^{-\alpha(g)} \right) \]

7. $\beta > \alpha(g)$
   \[ A(n) = O\left(n^{-\alpha(g)} \right) \]

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These 7 formulas are not sharp since it is possible to go further in the asymptotic expansion but it is not necessary here since one just needs to see the shape of the formulas which will be paradoxically sharp in part II. For instance if we consider the special case $\beta = 1 (> \alpha (g))$ we get from 1.2.2. the more precise estimate

$$A(n) \sim \frac{1}{\Gamma (2 - \alpha (g))} n^{-\alpha (g)} \quad (n \to \infty)$$

### 3.3.1 Proof of theorem 3.3

Without loss of generality I take $c_1 = c_0 = 1/2$ so that $\alpha (g) = 1/2$ and will prove the formula 2 in 2.3. (that is for the case $\beta = -1/2$). The other formulas are proved similarly for any $c_1, c_0 > 0$ and any $\beta$. So let

- $g(x) = \frac{x}{2} + \frac{1}{2}$
- $A_g(n) = n^{1/2}$
- $h(n) = n^{-1} (n^{3/2} - (n - 1)^{3/2})$

First using the lemma we get

$$A(n) = h(n) + \frac{(1/2)n}{n!} \left( 2 + \sum_{k=2}^{n-1} h(k) \frac{k!}{(1/2)_k} \right)$$

Next it is easy to see that we have the 3 asymptotic formulas as $n \to \infty$

$$h(n) = \frac{3}{2} n^{-1/2} - \frac{3}{8} n^{-3/2} + O \left( n^{-5/2} \right)$$

$$\Gamma \left( 1/2 \right) \frac{1/2 n}{n!} = n^{-1/2} - \frac{1}{8} n^{-3/2} + O \left( n^{-5/2} \right)$$

$$\frac{1}{\Gamma \left( 1/2 \right)} h(k) \frac{k!}{(1/2)_k} = \frac{3}{2} - \frac{3}{16} n^{-1} + O \left( n^{-2} \right)$$

therefore we get

$$A(n) = O \left( n^{-1/2} \right) + \left( n^{-1/2} - \frac{1}{8} n^{-3/2} + O \left( n^{-5/2} \right) \right) \left( O(1) + \sum_{k=2}^{n-1} \left( \frac{3}{2} - \frac{3}{16} k^{-1} \right) \right)$$

yielding

$$A(n) = \frac{3}{2} n^{1/2} - \frac{3}{16} n^{-1/2} \log n + O \left( n^{-1/2} \right)$$

$\square$. 

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3.3.2 Remark on the factor $-\frac{1}{g^*(-\beta)}$

It seems interesting to see how the factor $-\frac{1}{g^*(-\beta)}$ arises using Riemann integral when $\beta < 0$ and not using the formula 3 in 1.2.2. Indeed in the sequel this factor will appear in many settings and this appearance is a well known fact in tauberian theory using the classical Mellin transform and a change of variable (see for instance [Bin2] where arithmetic sums over the primes are considered).

Here from the condition $c_1, c_0 > 0$ we have

- $\alpha = \frac{c_0}{c_0 + c_1} \in [0, 1[$

Next it is easy to see that letting $A_g(n) = n^{-\beta}$ and $\beta \leq 0$ we have $a_n > 0$ for $n$ large enough. Now from theorem 1 we get

- $\beta < 0 < \alpha \Rightarrow A(n) \sim Cn^{-\beta} (n \to \infty)$ for a constant $C > 0$

Thus if $\beta < 0$ we have $a_n > 0$ which decreases monotonically to zero and $A(n) \to \infty$ as $n \to \infty$ hence we can infer

$$a_n = A(n) - A(n-1) \sim C(-\beta)n^{-\beta - 1} (n \to \infty)$$

so that plugging $a_k = C(-\beta)k^{-\beta - 1}$ in $A_g(n) = \sum_{k=1}^{n} a_k g\left(\frac{k}{n}\right)$ we get

$$A_g(n) = n^{-\beta} \sim C(-\beta) \sum_{k=1}^{n} k^{-\beta - 1} g\left(\frac{k}{n}\right) (n \to \infty)$$

yielding from the fact that $t^{-\beta - 1}g(t)$ is Riemann integrable on $[0, 1]$

$$\frac{-1}{C\beta} \sim \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{-\beta - 1} g\left(\frac{k}{n}\right) (n \to \infty) \sim \int_{0}^{1} t^{-\beta - 1}g(t)dt = g^*(-\beta)$$

hence we get

$$C = \frac{-1}{\beta g^*(-\beta)}$$

The interest of this method for $\beta < 0$ using Riemann integral is that it should work for any FGV, not only affine functions, and as we shall see this factor will arise in many subsequent tauberian formulas.

4 Polynomials are FGV

In this section it is proved that polynomials are FGV satisfying the primary definition of a function of good variation. The proof of this theorem is not a constructive one and is just sketched out.
4.1 Theorem
Let
\[ g(x) = \sum_{j=0}^{m} c_j x^j \]
where \( m \geq 1 \) and \( c_m g(0) g(1) \neq 0 \). Then \( g \) is a FGV satisfying the primary definition of index
\[ \alpha (g) = \min \left\{ \Re (\rho) \mid \sum_{j=0}^{m} \frac{c_j}{\rho - j} = 0 \right\} \]

4.2 Proof of theorem 4.1
Let \( A_g(n) = n^{-\beta} \) so that we have
\[ \sum_{k=1}^{n} a_k \sum_{j=0}^{m} c_j \left( \frac{k}{n} \right)^j = n^{-\beta} \tag{1} \]

Now multiplying (1) by \( n^m \) and using the forward difference operator \( \Delta \) \( m \) times we get (details omitted)
\[ \sum_{j=0}^{m} P_j(n) \Delta^{(j)}(A(n)) = \Delta^{(m)}(n^{m-\beta}) \tag{2} \]
where \( P_j \) are polynomials of degree \( j \).

(2) is a difference equation analogue to an Euler-Cauchy differential equation and to solve \( A(n) \) as \( n \to \infty \) requires to plug \( A(n) = n^{-\rho} \) in (2). But instead of doing that let us transform (1) as follows
\[ \sum_{j=0}^{m} c_j n^{-j} \sum_{k=1}^{n} a_k k^j = n^{-\beta} \tag{3} \]

Letting \( A(x) = \sum_{1 \leq k \leq x} a_k \) and using the well known formula
\[ \sum_{y < k \leq x} a_k f(k) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt \]
we get for any \( j \geq 1 \)
\[ \sum_{k=1}^{n} a_k k^j = A(n)n^j - j \int_0^n A(t)t^{j-1}dt \]
whence (3) becomes
\[ \left( \sum_{j=0}^{m} c_j \right) A(n) - \sum_{j=1}^{m} j c_j n^{-j} \int_{0}^{n} A(t)t^{j-1} dt = n^{-\beta} \] \quad (4)

so that taking \( A(x) = x^{-\rho} \) with the RHS of (4) equal to zero we get

\[ \left( \sum_{j=0}^{m} c_j \right) - \sum_{j=1}^{m} \frac{j c_j}{j - \rho} = 0 \]

From the condition \( c_0 \neq 0 \) we see that \( \rho \) can’t be zero and so we have to solve

\[ \sum_{j=0}^{m} \frac{c_j}{j - \rho} = 0 \]

Hence letting

\[ \alpha (g) = \min \left\{ \Re (\rho) \mid \sum_{j=0}^{m} \frac{c_j}{\rho - j} = 0 \right\} \]

we have from the difference equation (2) and arguments in 1.3. letting \( g^*(z) = \sum_{j=0}^{m} \frac{c_j}{z^j} \)

\[ \beta < \alpha (g) \Rightarrow A(n) \sim \left( \frac{-1}{\beta g^*(\beta)} \right) n^{-\beta} \quad (n \to \infty) \]

and

\[ \beta \geq \alpha (g) \Rightarrow A(n) \ll n^{-\alpha(g)} (\log n)^{b-1} \]

where \( b \) is the largest order of multiplicity of all root \( \rho \) satisfying \( \Re (\rho) = \alpha (g) \).

Hence \( g \) is a FGV satisfying the primary definition of index \( \alpha (g) \). □

4.3 A comparison conjecture for index of polynomials

It is worth to mention this conjecture which was checked for polynomials taken randomly up to degree 30.

4.3.1 Conjecture

Let \( P, Q \) be 2 polynomials with positive coefficients. Then we have

\[ \alpha (PQ) \leq \min (\alpha (P), \alpha (Q)) \]
4.3.2 Experimental support

The table belows illustrates this conjecture.

<table>
<thead>
<tr>
<th>$P(x)$</th>
<th>$Q(x)$</th>
<th>$\alpha (P)$</th>
<th>$\alpha (Q)$</th>
<th>$\alpha (PQ)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + 1$</td>
<td>$2x + 1$</td>
<td>0.5000</td>
<td>0.3333</td>
<td>0.2046</td>
</tr>
<tr>
<td>$x + 1$</td>
<td>$x^2 + x + 1$</td>
<td>0.5000</td>
<td>0.4226</td>
<td>0.2416</td>
</tr>
<tr>
<td>$x + 1$</td>
<td>$x^3 + x^2 + x + 1$</td>
<td>0.5000</td>
<td>0.3819</td>
<td>0.2150</td>
</tr>
<tr>
<td>$x^2 + x + 1$</td>
<td>$x^2 + 1$</td>
<td>0.4226</td>
<td>1.0000</td>
<td>0.3063</td>
</tr>
<tr>
<td>$x^2 + x + 1$</td>
<td>$x^2 + 3x + 1$</td>
<td>0.4226</td>
<td>0.2254</td>
<td>0.1135</td>
</tr>
<tr>
<td>$x^3 + x + 1$</td>
<td>$x^3 + x^2 + x + 1001$</td>
<td>0.4226</td>
<td>0.9990</td>
<td>0.4221</td>
</tr>
<tr>
<td>$x^3 + x + 1$</td>
<td>$x^3 + 1$</td>
<td>0.4514</td>
<td>1.5000</td>
<td>0.3605</td>
</tr>
<tr>
<td>$x^3 + x + 1$</td>
<td>$x^3 + 1$</td>
<td>0.4514</td>
<td>2.0000</td>
<td>0.3793</td>
</tr>
<tr>
<td>$x^3 + x + 1$</td>
<td>$x^3 + x^2 + x^2 + x + 1$</td>
<td>0.4514</td>
<td>0.3365</td>
<td>0.1535</td>
</tr>
<tr>
<td>$x^4 + 30x^3 + 10x^2 + 100x + 57$</td>
<td>$x^4 + 1$</td>
<td>0.3380</td>
<td>2.0000</td>
<td>0.2850</td>
</tr>
<tr>
<td>$x^4 + 30x^3 + 10x^2 + 100x + 57$</td>
<td>$x^3 + x^2 + x^2 + x + 1$</td>
<td>0.3380</td>
<td>0.3365</td>
<td>0.1233</td>
</tr>
<tr>
<td>$x^4 + 30x^3 + 10x^2 + 100x + 57$</td>
<td>$x^3 + x^2 + 100x + 1000$</td>
<td>0.3380</td>
<td>0.9090</td>
<td>0.3175</td>
</tr>
<tr>
<td>$x^5 + x^4 + 100x + 1000$</td>
<td>$x^3 + x^3 + 100x + 1000$</td>
<td>0.9090</td>
<td>0.9090</td>
<td>0.1406</td>
</tr>
</tbody>
</table>

4.3.3 Remark

There is equality if one of the polynomials is constant and I think it is the only case. Taking the simplest case of polynomials of degree 1

- $P(x) = x + a$ and $Q(x) = x + b$ for $a > 0, b > 0$

the conjecture $\min (\alpha (P), \alpha (Q)) \geq \alpha (PQ)$ is true since it is easy to show that we have the cumbersome inequality

$$3ab + 2a + 2b + 1 - \sqrt{a^2b^2 + 4a^2b + 4ab^2 + 4a^2 + 4b^2 + 6ab + 4a + 4b + 1} < 2 \min (a(b + 1), b(a + 1))$$

5 Analytic conjecture for continuous functions

The theorem 4.1 extends naturally to continuous functions yielding the following analytic conjecture for continuous functions.

5.1 Conjecture

Let $g$ be $C^k$ on $[0, 1]$ with $k \geq 0$ such that $g(0)g(1) \neq 0$ and define for $\Re z < 0$

the little Mellin transform of $g$ as follows

$$g^*(z) = \int_0^1 g(t)t^{-z-1}dt$$
Suppose $g^*$ can be continued analytically with possibly some singularities and let

$$\alpha(g) = \inf \{\Re(\rho) \mid \rho \in \mathbb{C} \land g^*(\rho) = 0\}$$

Then $g$ is a FGV of index $\alpha(g)$ according to the primary definition of FGV and if $A_g(n) = n^{-\beta}$ we get

$$\beta < \alpha(g) \Rightarrow A(n) \sim \left(\frac{-1}{\beta g^*(\beta)}\right) n^{-\beta} \quad (n \to \infty)$$

$$\beta \geq \alpha(g) \Rightarrow A(n) \ll n^{-\alpha(g)} L(n)$$

where $L$ is slowly varying. Moreover if $\alpha(g)$ is a real root of $g^*$ of order $b \geq 1$ we have

$$\beta = \alpha(g) \Rightarrow A(n) \sim C n^{-\alpha(g)} (\log n)^{b-1} \quad (n \to \infty)$$

$$\beta > \alpha(g) \Rightarrow A(n) \sim C(\beta) n^{-\alpha(g)} \quad (n \to \infty)$$

where $C, C(\beta) \neq 0$ are constants.

### 5.2 Remark

The conjecture is almost proved using the theorem 3 and the Weierstrass approximation theorem, i.e. for any $\varepsilon > 0$ there exists a polynomial $P_\varepsilon$ such that we have

$$\forall k \in \{1, 2, \ldots, n\}, \quad \left| g\left(\frac{k}{n}\right) - P_\varepsilon\left(\frac{k}{n}\right) \right| < \varepsilon$$

and the fact that

$$\lim_{\varepsilon \to 0} \alpha(P_\varepsilon) = \lim_{\varepsilon \to 0} \left(\inf \{\Re(\rho) \mid \rho \in \mathbb{C} \land P_\varepsilon^*(\rho) = 0\}\right) = \inf \{\Re(\rho) \mid \rho \in \mathbb{C} \land g^*(\rho) = 0\}$$

### 5.3 Experimental support

In order to convince the reader of the consistency of the analytic conjecture for continuous functions, three examples are given below, among hundreds of experiments made in recent years an supporting clearly this conjecture.
5.3.1 Using a property of the sine function

A simple trick allows us to construct a continuous function $g$ generated by an infinite series with good variation index $\alpha(g) = \frac{1}{2}$. Namely let

$$g(x) = 1 + \frac{\pi}{2}x \cos \left( \frac{\pi}{2} \sqrt{x} \right) = 1 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{(2n-2)!} \left( \frac{\pi}{2} \right)^{2n-1} x^n$$

so that we have

$$g^*(z) = -\frac{1}{z} + \sum_{n \geq 1} \frac{(-1)^{n-1}}{(2n-2)!} \left( \frac{\pi}{2} \right)^{2n-1} \frac{1}{n-z}$$

hence we get

$$g^* \left( \frac{1}{2} \right) = -2 + 2 \sin \left( \frac{\pi}{2} \right) = 0$$

Some checks confirm that $g^*$ has no zero in the half-plane $\Re z < \frac{1}{2}$ whence from the analytic conjecture we have $\alpha(g) = \frac{1}{2}$. Taking $\beta = \alpha(g) - 1 = -\frac{1}{2}$ we can see that the formula given for the case 2 in 3.3. works here too.

$$A(n) - \frac{n^{0.5}}{0.5g^*(-0.5)} n^{1/2} \log n \mbox{ when } A_g(n) = n^{1/2}$$

It seems to converge toward a limit.
5.3.2 A simple $C^\infty$ example

Let

$$g(x) = \frac{2}{1 + x}$$

so that

$$g^*(z) = \psi_0 \left( \frac{1 - z}{2} \right) - \psi_0 \left( 1 - \frac{z}{2} \right) - \frac{2}{z}$$

where $\psi_0$ is the digamma function. The 2 zeros $\rho$ of $g^*$ with smallest real part are

$$\rho = 1.34652 \pm 1.05516...i$$

hence from the analytic conjecture we have

$$\alpha(g) = 1.34652...$$

Let see how experiments support this.

fig.7bis) $A(n)n^{1.34652}$ when $A_g(n) = 0$ (black) and $A_g(n) = n^{-1.34652}$ (red)

Both graph are bounded and oscillating since the zero of $g^*$ with the smallest real part is complex.
fig.8) \( A(n)n^{0.5} \) vs \( \frac{-1}{0.5g^*(0.5)} = 0.2800495 \ldots \) when \( A_g(n) = n^{-0.5} \)

The convergence toward \( \frac{-1}{0.5g^*(0.5)} \) is clear.

5.3.3 A \( C^0 \) example

Let

\[
g(x) = \left| x - \frac{1}{2} \right| + \frac{1}{2}
\]

so that

\[
g^*(z) = \frac{2z - z}{z(1 - z)}
\]

and the zero of \( g^* \) with smallest real part is \( 0.824678546 + 1.5674321 \ldots i \) hence from the analytic conjecture we have

\[
\alpha(g) = 0.8246785\ldots
\]

Hereafter experimental support is provided supporting this fact.
Again it is bounded and oscillating since the zero of \( g^* \) with smallest real part is complex.

The convergence toward \( \frac{-1}{0.5g^*(0.5)} = 0.54691816... \) is slow but clear.

5.4 Functions with a discontinuity at zero

It is interesting to take a look at functions discontinuous at zero but bounded and continuous on \([0, 1]\). Sometime they are \( FGV \) fitting the analytic conjecture and sometime they even seem not to be \( FGV \).
For instance if we take \( g(x) = x + \sin(\log x) \) we have \( g^*(z) = \frac{z(1+z)}{(1-z)(1+z^2)} \) and we suspect that \( g \) is a FGV of index \( \alpha(g) = -1 \) which looks supported by experiments.

However if we take \( g(x) = x + \cos(\log x) \) we have \( g^*(z) = \frac{2z^2+z+1}{(1-z)(1+z^2)} \) hence we expect to have \( \alpha(g) = 1/4 \) but this is not supported by experiments and \( g \) seems even not to be a FGV according to the primary definition.

So there is a room here for further investigations and bounded variation seems to be an ingredient.

6  The tauberian equivalence principle for continuous functions

There are trivial equivalence between sums and integrals. For instance if \( a_n \) is a sequence of positive terms such that \( a_n \sim n^\lambda \) \( (n \to \infty) \) where \( \lambda > -1 \) and \( f > 0 \) is a monotonic function satisfying \( f(x) \sim x^\lambda \) \( (x \to \infty) \) then we have

\[
\sum_{k=1}^{n} a_k \sim \int_0^n f(t)dt \sim \frac{n^{\lambda+1}}{\lambda + 1} \quad (n \to \infty)
\]

If we have less precise information on \( a \) and \( f \) it is difficult in general to state such an equivalence. However the theorem 3.3 and the analytic conjecture led me to write down the following corollary, the tauberian equivalence principle summarizing somewhat good variation theory for continuous functions. It makes clear the tauberian duality between discrete and continuous mathematics.

6.1 Conjecture

The interest of this principle is that in general it is easier to handle integrals and so we can obtain asymptotic informations on discrete sums.

Let \( g > 0 \) be continuous on \([0,1]\) and continuous-discrete equation

\[
\int_0^x f(t)g \left( \frac{t}{x} \right) dt = \sum_{1 \leq k \leq x} a_k g \left( \frac{k}{x} \right) = x^{-\beta}
\]

Then for any \( \beta < \alpha(g) \) I claim that we have

\[
\int_0^x f(t)dt \sim \sum_{1 \leq k \leq x} a_k \sim \left( \frac{-1}{\beta g^*(\beta)} \right) x^{-\beta} \quad (x \to \infty)
\]

6.2 Remark

In the APPENDIX 1 I provide an autonomous proof of this tauberian equivalence principle for polynomials of degree 1 in order to convince the reader of the consistency of this principle which will be considerably extended later.
7 On functions discontinuous at $\frac{1}{2}$

Before the part II it seems interesting to show that some discontinuous functions almost constant on $[0, 1]$ with a single discontinuity at $\frac{1}{2}$ are FGV according to the primary definition of FGV. This theorem relies mainly on real analysis since any extension of the analytic conjecture would be completely useless here. This underlines that good variation theory has deep roots in real analysis too and complex analysis is sometime superfluous.

7.1 Theorem

For $0 < r < 1$ let us define the function $g$ by

- $g(x) = r$ if $x = \frac{1}{2}$
- $g(x) = 1$ if $x \neq \frac{1}{2}$

Then $g$ is a FGV of index

$$\alpha(g) = -\frac{\log(1 - r)}{\log(2)}$$

satisfying the 2 tauberian properties letting $A_g(n) = n^{-\beta}$

- $\beta < \alpha(g) \Rightarrow A(n) \sim n^{-\beta} (n \to \infty)$
- $\beta \geq \alpha(g) \Rightarrow A(n) = O(n^{-\alpha(g)})$

and more precisely we have the 4 sharp asymptotic formulas

1. $\beta < \alpha(g) - 1$
   $$A(n) = n^{-\beta} + O(n^{-1-\beta})$$

2. $\beta = \alpha(g) - 1$
   $$A(n) = n^{-\beta} + O\left(n^{-\alpha(g)} \log n \right)$$

3. $\alpha(g) - 1 < \beta < \alpha(g)$
   $$A(n) = n^{-\beta} + O\left(n^{-\alpha(g)} \right)$$

4. $\beta \geq \alpha(g)$
   $$A(n) = O\left(n^{-\alpha(g)} \right)$$

where again $\beta = \alpha(g) - 1$ is a crucial breakpoint.
7.2 Sketch of proof of theorem 7.1

From the definition of $g$ we have for $n \geq 1$

- $A_g(2n - 1) = A(2n - 1)$
- $A_g(2n) = A(2n) - (1 - r)a_n$

Therefore we have only to work out what $A(2n)$ is and if we set

$$v_n = A(2n) - (2n)^{-\beta}$$

we get $v(1) = 1 - r$ and for $n \geq 2$

$$v(n) = (-1)^n (1 - r)v\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + (1 - r)\left(n^{-\beta} - (n - 1)^{-\beta}\right)$$

and using this simple recursive formula one obtains the 4 above asymptotic formulas.

7.3 Remarks

The case $r > 1$ can also be considered and then $g$ is a FGV of index

$$\alpha (g) = -\frac{\log |1 - r|}{\log(2)}$$

7.4 Illustration of the different cases

We take $r = 1 - \frac{1}{\sqrt{2}}$ so that from theorem 3 we get $\alpha (g) = \frac{1}{2}$. In the following graphics one can see fractal patterns due to the discontinuity at $\frac{1}{2}$ which become clearer when $\beta > 0$. 
fig. 11) \( A(2n) - (2n)^{-\beta} \) \( (2n)^{1+\beta} \) for \( \beta = -0.75 \)

fig. 12) \( A(2n) - (2n)^{-b} \) \( (2n)^{1/2} \) \( \log(2n) \) for \( \beta = -\frac{1}{2} \)
fig.13) $(A(2n) - (2n)^{-\beta}) (2n)^{1/2}$ for $\beta = -0.25$

fig.14) $(A(2n) - (2n)^{-\beta}) (2n)^{1/2}$ for $\beta = +0.25$
fig.15) \( A(2n) - (2n)^{-\beta} \) \( (2n)^{1/2} \) for \( \beta = +0.50 \)

fig.16) \( A(2n) - (2n)^{-\beta} \) \( (2n)^{1/2} \) for \( \beta = +0.75 \)
Part II

Broken harmonic functions

The theorems in part I are not pure tauberian theorems since in fact no condition is required on the behaviour of $a_n$. The recursion and the smoothness of the weight function $g$ force $a_n$ to behave nicely so that the Hardy-Littlewood condition $a_n = O \left( \frac{1}{n} \right)$ is hidden. Hence they could bear the name of mercerian theorems and the analytic conjecture for continuous functions could be considered as a mercerian conjecture too.

What makes the theory really interesting from a number theoretic and tauberian viewpoint relies on the so called broken harmonic functions (BHF) which are discontinuous functions having infinitely many discontinuities. Indeed here one needs a tauberian condition on $a_n$, the Hardy-Littlewood-Ramanujan criterion (HLR criterion), in order to make an analytic conjecture similar to the analytic conjecture for continuous functions.

8 Introduction to part II

The BHF satisfying the HLR criterion are FGV sharing apparently a set of common tauberian properties. In what follows these strong tauberian properties are described in detail via a set of conjectures and supported by experiments.

BHF and the HLR criterion are defined in section 9 where the analytic conjecture for BHF satisfying the HLR criterion is also formulated.

Then in section 10 we consider a family of peculiar BHF, namely the functions $g_\lambda$ defined by

$$g_\lambda(x) = x^\lambda \left\lfloor \frac{\log x}{\log \lambda} \right\rfloor$$

where $\lambda \geq 2$ is any integer value. They fit clearly the analytic conjecture and provide the first example of functions having infinitely many discontinuities which are FGV (conjecture 10.1).

In section 11 the conjecture 11.1 related to BHF satisfying the HLR criterion allows us to derive the prime number theorem as a limit case.

In section 12 the tauberian symmetry conjecture (conjecture 12.1) is formulated with precise tauberian formulas reminiscent of the formulas for the simple case of affine functions.

In section 13 other formulations are given.

The section 14 shows that RH can be deduced from the tauberian symmetry conjecture and the method is naturally extended to the grand Riemann hypothesis (GRH) [Sar].

The tauberian equivalence principle for BHF satisfying the HLR criterion is formulated in section 15 and relates discrete to continuous mathematics similarly as what we say in part I.

In section 16 the definition of FGV is generalised with a remainder term.
In section 17 I provide a proof to that the Ingham function satisfies the \textit{HLR} criterion (theorem 17.1) as well as an interesting conjecture. Then I provide experimental support to the fact that \textit{BHF} of interest satisfy this criterion while others don’t.

Section 18 is devoted to a \textit{FGV} having a single discontinuity which could help to understand better what is going on regarding the relationship between the zeros of the little Mellin transform and the \textit{HLR} criterion.

Then in section 19 I generalise the tauberian equivalence principle for \textit{FGV}.

In section 20 I recall a comparison conjecture (conjecture 20.1) between index of \textit{BHF} satisfying and not satisfying the \textit{HLR} criterion and consider a family of functions supporting the conjecture.

In the section 21 I describe how application of good variation could apply to a classical problem related to the mean of the Euler totient function.

9 Definitions and the analytic conjecture for \textit{BHF}

9.1 Definition of \textit{BHF}

These functions were introduced for the first time in \cite{Clo1}. A function \(g\) is a \textit{BHF} if it exists a real positive sequence \((u_n)_{n \geq 1}\) satisfying

\[
1 = u_1 > u_2 > u_3 > \ldots > u_\infty = 0
\]

and such that for any \(n \geq 1\) we have

\[
 u_{n+1} < x \leq u_n \Rightarrow g(x) = v_n x
\]

where \(v_n > 0\) is an increasing sequence of reals such that \(\forall i \geq 1\) we have \(|u_i v_i| < M\) for a constant \(M > 0\).

Examples

- The Ingham function \(\Phi(x) = x \left\lfloor \frac{1}{x} \right\rfloor\) with \(u_i = \frac{1}{i}\) and \(v_i = i\).
- For \(\lambda > 1\) \(g_\lambda(x) = x \lambda \left\lfloor -\frac{\log x}{\log \lambda} \right\rfloor\) with \(u_i = \frac{1}{\lambda^{i-1}}\) and \(v_i = \lambda^{i-1}\)
- The generalised Ingham functions defined by

\[
\Phi_\chi(x) = x \sum_{1 \leq k \leq 1/x} \chi(k) \left\lfloor \frac{1}{kx} \right\rfloor
\]

with \(u_i = \frac{1}{i}\) and \(v_i\) is an arithmetical function depending on \(\chi\).
9.2 Definition of the HLR criterion

A function $g$ satisfies the HLR criterion if for any $\beta \geq 0$ we have the property

$$A_g(n) = n^{-\beta} \Rightarrow \forall \varepsilon > 0 \lim_{n \to \infty} a(n)n^{1-\varepsilon} = 0$$

This criterion appears to be fundamental. The name comes from Hardy-Littlewood condition in their first tauberian theorems [ChQ, Kor] and from a conjecture of Ramanujan on the size of coefficients of Dirichlet series in the Selberg class [Sel2]. This last point is well illustrated in section 17 using the Ramanujan $\tau$ function where it is proved that interesting $FGV$ satisfy this criterion.

9.3 Definition of the little Mellin transform

We recall the definition of the little Mellin transform $g^*$ of a bounded and measurable function $g : [0, 1] \to \mathbb{R}$. Namely for $\Re z < 0$ it is

$$g^*(z) = \int_0^1 g(t)t^{-z-1}dt$$

With these definitions in mind the analytic conjecture for $BHF$ satisfying the HLR criterion is formulated thereafter.

9.4 Analytic conjecture for $BHF$

Let $g$ be a $BHF$ satisfying the HLR criterion. If the little Mellin transform $g^*$ can be continued analytically with possibly some singularities then $g$ is a $FGV$ of index $\alpha(g)$ given by

$$\alpha(g) = \inf \{ \Re(\rho) \mid \rho \in \mathbb{C} \land g^*(\rho) = 0 \}$$

9.5 Remark

Of course functions having finitely many discontinuities and satisfying the HLR criterion were also considered and are conjectured to obey the analytic conjecture. An example is provided in section 12.

Many other types of discontinuous functions with finitely or infinitely many discontinuities were also considered such as

$$g(x) = \Phi(x)^r$$

with $0 < r < 1$ which satisfy clearly the HLR criterion, fit the analytic conjecture and moreover we suspect that we have

$$\alpha(\Phi^r) = 1 - r$$

For instance let $A_{\Phi^{0.25}}(n) = n^{-1/2}$ then $A(n)n^{1/2}$ should converge since we expect to have $\alpha(\Phi^{0.25}) = 0.75$ (see below).
fig.17) $A(n)n^{1/2}$ with $A_{\Phi \approx 0.25}(n) = n^{-1/2}$

It is clearly bounded and the convergence is probable (although it is very slow and erratic due to the fact that $1/2$ is close to $3/4$). The convergence is better seen with the following computation.

fig.18) $A(n)n^{1/4}$ with $A_{\Phi \approx 0.25}(n) = n^{-1/4}$

However a focus on $BHF$ is made here in order to show the relevance of the good variation theory regarding $RH$ and to keep the note not too long.
10 Tauberian properties of the functions $g_{\lambda}$

These functions were crucial to help me to define accurately good variation theory for nontrivial functions. Indeed they are not trivial but easier to handle than the Ingham function and indicate that a tauberian condition should be satisfied by $a_n$ to formulate an analytic conjecture similar to the analytic conjecture for continuous functions. It is the study of these functions which gave rise to the precise concept of the HLR criterion.

10.1 Conjecture

Let $\lambda \geq 2$ be an integer value and define

$$g_{\lambda}(x) = x^{\lambda \left\lfloor \frac{\log x}{\log \lambda} \right\rfloor}$$

then $g_{\lambda}$ is a BHF satisfying clearly the HLR criterion (see section 12) and we have

$$g_{\lambda}^*(z) = \frac{1 - \lambda^{z-1}}{(1 - z)(1 - \lambda z)}$$

so that from the analytic conjecture for BHF we should have $\alpha(g_{\lambda}) = 1$ since zeros are $\rho_k = 1 \pm \frac{2k\pi i}{\log \lambda}$ with $k \in \mathbb{Z}$.

Actually much more can be said since there are precise estimates for $A(n)$ depending on $\beta$ and looking like what was described in 1.3. for affine functions.

Namely if $A_{g}(n) = n^{-\beta}$ we have 4 cases yielding the following sharp asymptotic formulas

1. $\beta < 0$

$$A(n) = \left(\frac{-1}{\beta g_{\lambda}^*(\beta)}\right) n^{-\beta} + O\left(n^{-1-\beta}\right)$$

2. $\beta = 0$

$$A(n) = \frac{\lambda \log \lambda}{\lambda - 1} + O\left(n^{-1} \log n\right)$$

3. $0 < \beta < 1$

$$A(n) = \left(\frac{-1}{\beta g_{\lambda}^*(\beta)}\right) n^{-\beta} + O\left(n^{-1}\right)$$

4. $1 \leq \beta$

$$A(n) = O\left(n^{-1}\right)$$
10.2 Experimental support

I provide Experimental support for \( g_2 \) with the 3 graphics below which are clearly bounded.

\[
\text{fig.19) } A_{g_2}(n) = n^{-2} \text{ plot of } A(n)n
\]

\[
\text{fig.20) } A_{g_2}(n) = n^{-0.5} \text{ plot of } n \left( A(n) + \frac{n^{-0.5}}{0.5g_2^{-0.5}} \right)
\]
11 A conjecture yielding the PNT

The relevance of the following general conjecture for BHF satisfying the HLR criterion is to put the PNT within tauberian theory using good variation theory as a limit case when $\beta \to 0^-$. It is somewhat reminiscent of classical tauberian proof of the PNT [Kol] but seems easier to handle and could lead to a simpler proof than all known proofs of the PNT.

11.1 Conjecture

Let $g$ be a BHF satisfying the HLR criterion. Suppose that we have

- $\lim_{z \to 0} z g^*(z) = \ell \neq 0$ exists.

Then if for $n \geq n_0$ $A_g(n) = n^{-\beta}$ we get the two tauberian properties

$$-1 < \beta < 0 \Rightarrow A(n) = \left( -\frac{1}{\beta g^*(\beta)} \right) n^{-\beta} + o(1)$$

$$\beta = 0 \Rightarrow A(n) = -\frac{1}{\ell} + o(1)$$

11.2 Deriving the PNT

The PNT is a consequence of the conjecture 11.1.

---

3If $\lim_{x \to 0} g(x) = g(0) \neq 0$ exists then $\lim_{z \to 0} zg^*(z) = -g(0)$. 

11.2.1 Proof that the PNT is derived from the conjecture 11.1

Let the sequence \( w \) be defined by 
\[
\begin{align*}
\bullet & \quad w_{2n} = \mu_{2n} - \mu_n \\
\bullet & \quad w_{2n+1} = \mu_{2n+1}
\end{align*}
\]
where \( \mu \) is the Möbius function (for \( n \geq 2 \) \( w \) is the sequence A092673 in [Slo]), then it is easy to see that we have for \( n \geq 2 \)
\[
\sum_{k=1}^{n} \frac{w_k}{k} \Phi\left(\frac{k}{n}\right) = 1
\]
Next we expect that \( \Phi \) satisfies the HLR criterion and since we have
\[
\begin{align*}
\bullet & \quad \Phi^*(z) = \frac{\zeta(1-z)}{1-z} \\
\bullet & \quad \lim_{z \to 0} z \Phi^*(z) = -1
\end{align*}
\]
the conditions of the conjecture 2 are satisfied and we get
\[
\sum_{k=1}^{n} \frac{w_k}{k} = 1 + o(1)
\]
Then letting \( \mu'_1 = 1 \) and \( \mu'_n = w_n \) for \( n \geq 2 \) we have
\[
\sum_{k=1}^{n} \frac{\mu'_k}{k} = o(1)
\]
which yields by Abel summation
\[
\sum_{k=1}^{n} \mu'_k = o(n) \tag{5}
\]
Now let \( b \) denotes the sequence defined by
\[
\begin{align*}
\bullet & \quad b(n) = 1 \text{ if } n \text{ is a power of 2 and } b(n) = 0 \text{ otherwise}
\end{align*}
\]
then it is easy to see that we have
\[
\sum_{n \geq 1} \frac{b(n)}{n^s} \sum_{n \geq 1} \frac{\mu'_n}{n^s} = \frac{1}{\zeta(s)}
\]
which yields thanks to (5) and the fact that \( b(n) \geq 0 \)
\[
\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} b(i) \sum_{j=1}^{\lfloor n/i \rfloor} \mu'_j = o \left( \sum_{i=1}^{n} \frac{b(i)}{i} \right)
\]
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and thanks to the fact that \( \sum_{i=1}^{\infty} \frac{k(i)}{i} = \sum_{k=0}^{\infty} \frac{1}{2^k} = 2 \) we have finally

\[
\sum_{i=1}^{n} \mu_i = o(n)
\]

which is a well known equivalence to the PNT (see for instance \( \text{[Ing]}, \text{[Bor], [Ten]} \)).

11.2.2 Remark

The main difference between this derivation of the PNT and other classical tauberian theorems like the Ikehara’s theorem (cf. \( \text{[Ten]} \) p. 326) is that one can go further considering \( \beta > 0 \) where a new tauberian landscape emerges. This is described in the following conjecture.

12 Tauberian symmetry conjecture for BHF

The conjecture 2 is a kind of continuity conjecture when \( \beta \to 0 \) which can be made more precise recalling the conjecture 1 where \( 1 - \alpha (g_\lambda) = 0 \) as soon as \( \lambda \geq 2 \) is an integer value. Indeed it looks like the exponent in the error term for the tauberian formulas in 10.1 is a continuous function of \( \beta \) similar to what happens in the theorem 2 in section 3 part I for affine functions specially when \( \beta \to \alpha (g_\lambda) - 1 \). A bunch of experiments confirmed this fact for the Ingham function assuming that \( \alpha (\Phi) = \frac{1}{2} \). Thus it was quite apparent that a general conjecture can be formulated. It is the tauberian symmetry conjecture for BHF. This conjecture is perhaps the most important one regarding RH since it relates intrinsic properties of \( g \) (the HLR criterion) and properties of \( g^* \) (the functional equation) to tauberian properties of \( g \). In other words the symmetry of the functional equation is reflected in a tauberian symmetry if and only if RH is true.

12.1 Conjecture

Let \( g \) be a BHF satisfying the HLR criterion and suppose that

- \( \lim_{x \to 0} g(x) \neq 0 \) exists.
- there is a gamma factor \( G \) and a constant \( C \) such that we have the riemannian functional equation

\[
(1 - z)g^*(z)G(z) = Czg^*(1 - z)G(1 - z)
\]

Then \( \beta = \alpha (g) - 1 \) is still a breakpoint (i.e. like for the remainder term in tauberian formulas 2 in theorem 3.3 or in conjecture 10.1) so that I conjecture that the riemannian functional equation forces the intervall

\[
[\alpha (g) - 1, \alpha (g)]
\]
to be symmetrical around zero which implies

\[ \alpha(g) = \frac{1}{2} \]

and letting \( A_g(n) = n^{-\beta} \) we have the following 6 tauberian formulas covering all possible cases:

1. \( \beta < -\frac{1}{2} \)

\[ A(n) = \left( \frac{-1}{\beta g^*(\beta)} \right) n^{-\beta} + O \left( n^{-1-\beta} (\log n)^\varepsilon \right) \]

2. \( \beta = -\frac{1}{2} \)

\[ A(n) = \left( \frac{-1}{\beta g^*(\beta)} \right) n^{-\beta} + O \left( n^{-1/2} (\log n)^{1+\varepsilon} \right) \]

3. \( -\frac{1}{2} < \beta < 0 \)

\[ A(n) = \left( \frac{-1}{\beta g^*(\beta)} \right) n^{-\beta} + O \left( n^{-1/2} (\log n)^\varepsilon \right) \]

4. \( \beta = 0 \)

\[ A(n) = -\frac{1}{\ell} + O \left( n^{-1/2} (\log n)^\varepsilon \right) \]

5. \( 0 < \beta < \frac{1}{2} \)

\[ A(n) = \left( \frac{-1}{\beta g^*(\beta)} \right) n^{-\beta} + O \left( n^{-1/2} (\log n)^\varepsilon \right) \]

6. \( \beta \geq \frac{1}{2} \)

\[ A(n) = O \left( n^{-1/2} (\log n)^\varepsilon \right) \]

**Remark**

Since zeros are necessarily on the critical line it is almost like \( g \) has a single real zero. But the influence of nontrivial zeros on the error term can’t be neglected. Hence the tauberian formulas with remainder term of the theorem 3.3. can be replicated providing that we add a slowly varying function as a factor and here I claim that \( \log(n)^\varepsilon \) works. Indeed I need to ensure consistency with expected properties of nontrivial zeros such as the linear independence over the rationals. More precisely if the non trivial zeros of \( g^* \) are dependant over the rationals (which is the case for the zeros of \( g^*_2 \) considered in 10.2) we don’t need to consider this factor as with the formulas in the conjecture 10.1.

For \( g = \Phi \) however we need for instance to be in accordance with a theorem of Ingham \[ Ing \] [Gro] [Saf] relating the asymptotic behaviour of the summatory functions of the Möbius function and of the Liouville function to linear dependence properties of the non trivial zeros over the rationals. Also some \( \Omega \) results
suggest that $\log(n)^{\varepsilon}$ is enough as shown in section 22 with a remainder term related to a summatory function involving the Euler totient function. It is likely that $\log(n)^{\varepsilon}$ is not optimal. According to several studies like [Kot], one should expect for $g = \Phi$ and the case $\beta = 1$ that $A(n) = O \left( n^{-1/2} (\log \log n) \right)$ works. In general I suspect that there is an optimal slowly varying function $L_g$ depending on $g$ such we can replace $\log(n)^{\varepsilon}$ by $L_g(n)$ in each of the 6 above formulas and I guess that $L_\Phi(n) \ll \log \log n$.

12.2 Experimental support using the Ingham function

12.2.1 The case $\beta = -3/4$

Consider the case $\beta < -\frac{1}{2}$ and take $\beta = -0.75$. Then if $a_n$ is given by $A_\Phi(n) = n^{0.75}$ the previous conjecture yields

\[ n^{0.25} \left( A(n) + \left( \frac{7/3}{\zeta(7/4)} \right) n^{0.75} \right) = O(\log(n)^{\varepsilon}) \]

which is somewhat supported by the next graphic which looks bounded and could be bounded by $\log(n)^{\varepsilon}$.

fig. 22) Plot of $n^{0.25} \left( A(n) + \left( \frac{7/3}{\zeta(7/4)} \right) n^{0.75} \right)$ when $\beta = -0.75$

13 Other formulations

I give 2 other formulations which should be equivalent to the main aspect of the tauberian symmetry conjecture, the truth of $RH$.  

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13.1 The anti-\(HLR\) conjecture

Let \(g\) be a \(BHF\) which \(\text{doesn’t satisfy the } HLR\text{ criterion}\) and such that \((1 - z)g^*(z)\) can be continued analytically and satisfies a riemannian functional equation. Then \(g^*\) has infinitely many zeros on the line \(x = \frac{1}{2}\) but also infinitely many zeros off the critical line within the critical strip.

13.2 The pair of zeros conjecture

If \(g\) is a \(BHF\) such that \((1 - z)g^*(z)\) can be continued analytically, satisfies a riemannian functional equation and has \(\text{2 zeros on 2 distinct vertical lines}\) within the critical strip then \(g\) can not satisfy the \(HLR\) criterion. In other words if \(g\) is a \(BHF\) such that \((1 - z)g^*(z)\) can be continued analytically, satisfies a riemannian functional equation and the \(HLR\) criterion then all nontrivial zeros of \(g^*\) are on the critical line.

13.3 Summarising the previous conjectures

If \(g\) is a \(BHF\) satisfying \(g(0)g(1) \neq 0\) and:

- \(HLR\) means “\(g\) satisfies the \(HLR\) criterion”
- \(RFE\) means “\(zg^*(1 - z)\) satisfies a riemannian functional equation”
- \(RH\) means “\(zg^*(1 - z)\) satisfies the Riemann hypothesis”
- \(\overline{HLR}\) means “\(g\) doesn’t satisfy the \(HLR\) criterion”
- \(\overline{RFE}\) means “\(g^*\) doesn’t satisfy a riemannian functional equation”
- \(\overline{RH}\) means “\(zg^*(1 - z)\) doesn’t satisfy the Riemann hypothesis”

then we have the following 4 main properties:

1. \(HLR + RFE \Rightarrow RH\)
2. \(HLR + RFE \Rightarrow \overline{RH}\)
3. \(\overline{HLR} + RFE \Rightarrow \overline{RH}\)
4. \(\overline{RH} + RFE \Rightarrow \overline{HLR}\)

The fourth relation seems the more interesting for a good formulation of the problem. Indeed in order to prove that \(RH\) is true it would suffice to assume that \(RH\) is not true. Namely since \(\Phi\) satisfies the \(HLR\) criterion (see section 17) there would be a contradiction.
14 Corrolaries

14.1 RH is true

Since the Ingham function $\Phi$ satisfies the HLR criterion (see section 17) and $\Phi^\ast$ has a riemannian functional equation, the conditions of the tauberian symmetry conjecture are verified and we have $\alpha(\Phi) = \frac{1}{2}$ which is equivalent to RH.

14.2 The generalised RH is true

Since the generalised Ingham functions satisfy the HLR criterion (see section 17) RH is true for $L(1-z,\chi)$.

Proof

Let $\Phi_\chi(x) = x \sum_{1 \leq k \leq 1/x} \chi(k) \left\lfloor \frac{1}{kx} \right\rfloor$ be a generalised Ingham function, then we have (details ommitted)

$$\Phi^\ast_\chi(z) = \frac{\zeta(1-z)L(1-z,\chi)}{1-z}$$

which satisfies a riemannian functionnal equation and $\Phi_\chi$ satisfies the HLR criterion. Thus the conditions of the tauberian symmetry conjecture are verified and we have

$$\alpha(\Phi_\chi) = \frac{1}{2}$$

meaning that RH is true for $\zeta(1-z)L(1-z,\chi)$.

14.3 The Grand RH is true

The GRH is true assuming some BHF satisfy the HLR criterion.

Proof

Without loss of generality let consider the Ramanujan $\tau$ function (sequence A000594 in [Slo]) and the associated automorphic form. Let

$$g_\tau(x) = x \sum_{1 \leq k \leq 1/x} \frac{\tau_k}{k^{11/2}} \left\lfloor \frac{1}{kx} \right\rfloor$$

Since $g_\tau$ is a BHF satisfying the HLR criterion (see section 13), using the same arguments than above and the tauberian symmetry conjecture, RH is true for the corresponding Dirichlet series and its analytic continuation which satisfies a riemannian functional equation.
14.4 Experimental support for corrolary 14.2

Take the character $\chi = 1, 0, -1, 0, 1, 0, -1, 0, 1, ...$ then we have

$$\Phi^*_{\chi}(z) = \frac{\zeta(1-z)\beta(1-z)}{1-z}$$

where

$$\beta(s) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^s}$$

is the Dirichlet beta function. Then consider

$$A_{\Phi_{\chi}}(n) = n^{0.2}$$

under GRH one has

$$\alpha(\Phi_{\chi}) = \frac{1}{2}$$

and since $-\frac{1}{2} < -0.2 < 0$ from the tauberian symmetry conjecture we should have

$$A(n) = \left( \frac{1}{0.2\zeta(1.2)\beta(1.2)} \right) n^{0.2} + O \left( n^{-1/2} \log(n)^{\epsilon} \right)$$

The following graphic could indeed be bounded by $(\log n)^{\epsilon}$ supporting the tauberian symmetry conjecture.

fig.23) Plot of $n^{1/2} \left( A(n) - \left( \frac{1}{0.2\zeta(1.2)\beta(1.2)} \right) n^{0.2} \right)$
15 The tauberian equivalence principle for $BHF$

As in part I section 5 we can formulate a conjectured tauberian equivalence principle for $BHF$ satisfying the $HLR$ criterion.

Conjecture

Let $g$ be a $BHF$ satisfying the $HLR$ criterion and consider the continuous-discrete equation

$$\int_0^x f(t)g\left(\frac{t}{x}\right) dt = \sum_{1 \leq k \leq x} a_k g\left(\frac{k}{x}\right) = x^{-\beta}$$

Then for any $\beta < \alpha(g)$ we have

$$\int_0^x f(t)dt \sim \sum_{1 \leq k \leq x} a_k \sim \left(\frac{-1}{\beta g^*(\beta)}\right) x^{-\beta} \quad (x \to \infty)$$

16 Extension of the definition of $FGV$

In fact the primary definition of $FGV$ has to be extended in order to understand the asymptotic behaviour of sums like the summatory Liouville function [Bor2, Saf]. In order to avoid limitation problems for the Ingham summation method [Seg2] one has to be cautious in adding an extra term. So the following extension of the primary definition was made taking into account this caveat.

16.1 Definition of $FGV$ with a remainder term

Let $g$ be a bounded real function on $]0, 1]$ satisfying $g(1) \neq 0$ and let $h : \mathbb{R}^+ \to \mathbb{R}$ be any bounded function. Then $g$ is a $FGV$ of index

- $0 < \alpha(g) < 1$

if the following tauberian conditions are satisfied:

1. $\beta \leq 0 \land A_g(n) = n^{-\beta} + h(n)n^{-\beta-1} \Rightarrow A(n) \sim Cn^{-\beta} \quad (n \to \infty)$ with $C \neq 0$.

2. $0 < \beta < \alpha(g) \land A_g(n) = n^{-\beta} + h(n)n^{-1} \Rightarrow A(n) \sim Cn^{-\beta} \quad (n \to \infty)$ with $C \neq 0$.

3. $\alpha(g) \leq \beta < 1 \land A_g(n) = n^{-\beta} + h(n)n^{-1} \Rightarrow A(n) \ll n^{-\alpha(g)}L(n)$ where $L$ is slowly varying.

The interest of this definition is due to the fact that under $RH$ many asymptotic formulas for interesting summatory functions become available as described below in 16.3.
16.2 Conjecture for $BHF$ with a remainder term

Let $g$ be a $BHF$ satisfying the $HLR$ criterion of index $0 < \alpha(g) < 1$ then $g$ fits the definition 16.1 and satisfies the tauberian symmetry conjecture when $\beta > 0$.

16.3 On the Liouville function

The extension of the $FGV$ definition allows us to naturally get the asymptotic formula for the summatory Liouville function. Namely it is well known that

$$\sum_{k=1}^{n} \frac{(-1)^{\Omega(k)}}{k} \Phi\left(\frac{k}{n}\right) = n^{-1/2} - \frac{\{\sqrt{n}\}}{n}$$

where $(-1)^{\Omega(n)}$ is the Liouville function (sequence A008836 in [Slo]) then assuming $RH$ is true the conjecture 16.2. yields

$$\sum_{k=1}^{n} \frac{(-1)^{\Omega(k)}}{k} \ll n^{-1/2} \log(n)^\varepsilon$$

and then by Abel summation we have

$$\sum_{k=1}^{n} (-1)^{\Omega(k)} \ll n^{1/2} \log(n)^\varepsilon$$

which of course is a well known equivalence to $RH$ [Box].

16.4 Beyond the Liouville function

For a given real value $\beta$ let

$$b_n = (-1)^{\omega(n)} c_n$$

where $\omega(n)$ counts the prime factors of $n$ without multiplicity (sequence A001222 in [Slo]) and $c_n$ is the multiplicative function

$$c_{p^n} = \frac{1}{p^{(v-1)\beta}} \left(1 - \frac{1}{p^\beta}\right)$$

Then $b_n$ is bounded and its Dirichlet series is given by (details ommitted)

$$\frac{\zeta(s + \beta)}{\zeta(s)} = \sum_{n \geq 1} \frac{b_n}{n^s}$$

Next for any $n \geq 1$ it is easy to see that we have the identity

$$\sum_{k=1}^{n} b_k \frac{n}{k} = \sum_{k=1}^{n} k^{-\beta}$$
So that letting $a_n = (1 - \beta) \frac{k}{n}$ and using the asymptotic formula

$$\sum_{k=1}^{n} k^{-\beta} = \frac{1}{1 - \beta} n^{1-\beta} + \zeta(\beta) + \frac{1}{2} n^{-\beta} + O\left(n^{-\beta - 1}\right)$$

we get for $\beta \geq 0$

$$\sum_{k=1}^{n} a_k \Phi \left( \frac{k}{n} \right) = n^{-\beta} + (1 - \beta)\zeta(\beta) n^{-1} + o\left(n^{-1}\right)$$

and for $\beta < 0$

$$\sum_{k=1}^{n} a_k \Phi \left( \frac{k}{n} \right) = n^{-\beta} + \frac{1}{2} n^{-\beta-1} + o\left(n^{-\beta-1}\right)$$

and we can compute $a(n)$ efficiently up to big values of $n$. Hence one can check that $\Phi$ satisfies the conjecture 16.2. For instance the following graphic should be bounded by $\log(n)^7$ where $10^6$ terms of $a_n$ were computed.

fig.24) Plot of $\left( A(n) - \frac{n^{1/2}}{\zeta(3/2)} \right) \frac{n^{1/2}}{\log n}$ when $\beta = -1/2$
17 On the HLR criterion

As seen before this criterion is the cornerstone of this approach and I provide here a complete proof to the fact that the Ingham function satisfies this criterion. Next I provide experimental evidence for the functions $g_\lambda$, the generalised Ingham functions and the $L$ function associated to the Ramanujan tau numbers. Then I provide experiments supporting the fact that the BHF associated to the counter-example of Heilbronn doesn’t satisfy the HLR criterion.

17.1 Theorem

The Ingham function $\Phi$ satisfies the HLR criterion.

17.1.1 Proof of theorem 17.1

Let $A_g(n) = n^{-\beta}$ with $\beta \geq 0$ then we have

$$\sum_{k=1}^{n} a_k k \frac{\tau_1}{k} = \sum_{k=1}^{n} \sum_{d|k} a(d) d = n^{1-\beta} \Rightarrow \sum_{d|n} a(d) d = n^{1-\beta} - (n-1)^{1-\beta}$$

Case $\beta = 0$ We have simply $a_k k = w(k)$ the sequence considered in 11.2.1 which is bounded.

Case $\beta > 0$ So let us consider $\beta > 0$ so that $n^{1-\beta} - (n-1)^{1-\beta}$ is positive and decreases monotonically to zero as $n \to \infty$. Now by Moebius inversion formula we have

$$\sum_{d|n} a(d) d = n^{1-\beta} - (n-1)^{1-\beta} \Rightarrow a(n) n = \sum_{d|n} \mu \left( \frac{n}{d} \right) (d^{1-\beta} - (d-1)^{1-\beta})$$

whence we get

$$|a(n)| \leq \sum_{d|n} \left| \mu \left( \frac{n}{d} \right) (d^{1-\beta} - (d-1)^{1-\beta}) \right| \leq \sum_{k=1}^{\tau(n)} k^{1-\beta} - (k-1)^{1-\beta} = \tau(n)^{1-\beta} \ll \begin{cases} 1 & \beta \geq 1 \\ n^\epsilon & 0 < \beta < 1 \end{cases}$$

and the Ingham function satisfies the HLR criterion.

In fact there is something stronger to state since I claim that we have in fact $a(n) = O(n^{-1})$ in any case, which is the optimal Hardy-Littlewood tauberian condition. The following related conjecture illustrates this claim and is interesting on its own.
17.1.2 The minmax conjecture

In passing I mention this interesting conjecture for the behaviour of the record values of $na(n)$. Let $0 < \beta < 1$ and define the sequence $b$ as follows

- $\sum_{d|n} b(d) = n^{-\beta}$

so that

- $b(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^{-\beta}$

Let us define

- $M_\beta(x) = \max \{b(k) \mid k \leq x\}$
- $m_\beta(x) = \min \{b(k) \mid k \leq x\}$

Then I claim that we have the formulas

$0 < \beta < 1/2 \Rightarrow M_\beta(x) = 1 - 2x^{-\beta/2} + x^{-\beta} + O(x^{-2\beta})$

$1/2 \leq \beta < 1 \Rightarrow M_\beta(x) = 1 - 2x^{-\beta/2} + O(x^{-\beta})$

whereas

$m_\beta(x) = -1 + x^{-\beta} + O(x^{-1})$

17.1.3 Experimental support of the minmax conjecture

Case $0 < \beta < 1/2$

fig.25) $(1 - M_{0.3}(n) - 2n^{-0.15} + n^{-0.3}) n^{0.6}$ every 1000$n$

It looks bounded.
Case $1/2 \leq \beta < 1$

$$\text{fig.26) } \left( 1 - M_{0.7}(n) - 2n^{-0.35} \right) n^{0.7} \text{ every } 1000n$$

It looks bounded and there seems to be no convergence toward $-1$.

### 17.2 Experimental support for the functions $g_\lambda$

#### 17.2.1 Functions $g_\lambda$ satisfying the HLR criterion

Here we support the fact that the functions $g_\lambda$ studied in section 10 satisfy the HLR criterion whenever $\lambda \in \mathbb{N}^+$ and we take

$$g_2(x) = x^2 \left\lfloor \frac{\log x}{\log 2} \right\rfloor$$

$$\text{fig.27) Plot of } na_n \text{ taking } A_{g_2}(n) = n^{-1}$$

It is clearly bounded.
17.2.2 Functions $g_\lambda$ not satisfying the $HLR$ criterion

It is also interesting to see that $g_\lambda$ doesn’t satisfy the $HLR$ criterion when $\lambda > 1$ is not an integer value and thus the analytic conjecture doesn’t work. An important example is

$$g_{\sqrt{2}}(x) = x\sqrt{2} \left\lfloor -\frac{\log x}{\log \sqrt{2}} \right\rfloor$$

since I conjectured in a previous paper [Clo2] that

$$\alpha(g_{\sqrt{2}}) = \frac{1}{2}$$

not 1 and clearly $g_{\sqrt{2}}$ doesn’t satisfy the $HLR$ criterion as shown thereafter.

fig. 28) Plot of $na_n$ vs $\pm \frac{1}{3}\sqrt{n}$ (red) taking $A_{g_{\sqrt{2}}}(n) = n^{-1}$

It is bounded by something like $\sqrt{n}$. In an other hand it is worth to note that $g_{\sqrt{2}}$ fits the asymptotic formulas given in section 10 taking for instance $\beta = -1/2$. 

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fig. 29) Plot of $\frac{n^{0.5}}{\log n} \left( A(n) + \frac{n^{0.5}}{0.5 g(0.5)} \right)$ taking $A_{g}(n) = n^{0.5}$

It looks bounded.

17.3 The Ingham function

fig.30) Plot of $na_n$ taking $A_\Phi(n) = n^{-1}$

It is clearly bounded.
17.4 A generalised Ingham function

Let

\[ \chi = 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, -1, 0,... \]

and

\[ \Phi_\chi(x) = x \sum_{1 \leq k \leq 1/x} \chi(k) \left\lfloor \frac{1}{kx} \right\rfloor \]

Then the following graphic supports the fact that \( \Phi_\chi \) satisfies the HLR criterion.

![Plot of \( \Phi_\chi(n) \) for \( A_{g_\chi}(n) = n^{-1} \)]

It is clearly bounded so that \( \Phi_\chi \) satisfies the HLR criterion.

17.5 The BHF related to the Ramanujan tau function

Let

\[ g_\tau(x) = x \sum_{1 \leq k \leq 1/x} \frac{\tau_k}{k^{3/2}} \left\lfloor \frac{1}{kx} \right\rfloor \]

where \( \tau \) is the Ramanujan tau function. Then \( g_\tau \) is a BHF which should satisfy the HLR criterion.

Indeed computing 30000 Ramanujan tau numbers I was able to compute 30000 terms of \( a(n) \) given by the recursion
\[ A_{g_\tau}(n) = \frac{1}{2n} \]

It then seems that \( g_\tau \) satisfies the HLR criterion as shown by the graphic below where we compare \( na(n) \) to \( \pm 2\log n \) (red).

fig.32) Plot of \( na_n \) vs \( \pm 2\log n \) taking \( A_{g_\tau}(n) = n^{-1} \)

It looks like \( a(n)n \) stays of order \( \log n \) thus we would have \( \lim_{n \to \infty} a(n)n^{1-\varepsilon} = 0 \) for any \( \varepsilon > 0 \) and \( g_\tau \) would satisfy the HLR criterion.

Finally let see how the BHF associated to the Heilbronn-Davenport zeta function doesn’t satisfy the HLR criterion

17.6 The Heilbronn-Davenport zeta function

Letting

\[ \xi = \frac{-2 + \sqrt{10 - 2\sqrt{5}}}{\sqrt{5} - 1} = 0.284079... \]

(see sequence A158934 in Slo) Davenport and Heilbronn [Dav] considered the analytic continuation of the Dirichlet series

\[ H(s) = 1 + \frac{\xi}{2^s} - \frac{\xi}{3^s} - \frac{1}{4^s} + \frac{0}{5^s} + .... \]
and showed that despite the fact that $H$ has a riemannian functional equation it has nontrivial zeros off the critical line. In [San] many zeros of $H$ which are off the critical line are computed. Therefore it is interesting to consider

$$g_H(x) = x \sum_{1 \leq k \leq \lfloor \frac{1}{2} x \rfloor} h(k) \left\lfloor \frac{1}{kx} \right\rfloor$$

and see that it doesn’t satisfy the HLR criterion supporting for instance the conjectures formulated in 10.3 or 10.4.

fig.33) Plot of $na_n$ taking $A_{g_H}(n) = n^{-1}$ vs $\pm \sqrt{n}$ (red)

It is not bounded and it seems not bounded by any slowly varying function since some points $(n, na_n)$ are close to $(n, \pm \sqrt{n})$.

18 A FGV with a single discontinuity

Here a function having a single discontinuity is considered. This function satisfies clearly the HLR criterion and the analytic conjecture so that it could help to understand better what is going on, i.e. the reason why the HLR criterion for $BHF$ is related to the location of the zeros of the little Mellin transform.

Let

- $g(x) = x$ for $x > \frac{1}{2}$

and
• \( g(x) = x + \frac{1}{2} \) for \( 0 < x \leq \frac{1}{2} \)

which has a single discontinuity at \( x = \frac{1}{2} \) and continuous elsewhere.

Although \( g \) is not a BHF we will see that \( g \) is a FGV for which the analytic conjecture for BHF works which was not the case for the discontinuous functions with a single discontinuity considered in section 6 part I.

18.1 The HLR criterion is satisfied

First let us see that \( g \) satisfies the HLR criterion with the 2 following graphics.

fig.34) Plot of \( na_n \) taking \( A_g(n) = n^{-1} \)

fig.35 bis) Plot of \( na_n \) taking \( A_g(n) = n^{-2} \)
18.2 The analytic conjecture works

We have

\[ g^*(z) = \frac{1}{1 - z} - \frac{2^{z-1}}{z} \]

and the analytic conjecture is expected to work here where the zeros of \( g^* \) consist of a real zero \( \rho = 0.3970037789... \) and the complex zeros are in the half-plane \( \Re z > 1 \) as shown by the graphic below representing \( \arg (g^* (x + iy^3)) \) for \(-1 < x < 2 \) and \(-3.5 < y < +3.5\)

fig.36) \( \arg (g^* (x + iy^3)) \)

hence we get

\[ \alpha (g) = \inf \{ \Re (\rho) \mid \rho \in \mathbb{C} \land g^*(\rho) = 0 \} = 0.3970037789... \]

Therefore since \( \alpha (g) \) is a real zero of \( g^* \) we should have for instance the 3 following asymptotic formulas

\[ A_g(n) = \frac{1}{2^n} \Rightarrow A(n) \sim C n^{-0.3970037789} \quad (n \to \infty) \]

\[ A_g(n) = n^{-\alpha (g)} \Rightarrow A(n) \sim C' n^{-0.3970037789 \log n} \quad (n \to \infty) \]

\[ A_g(n) = n^{-0.1} \Rightarrow A(n) \sim C'' n^{-0.1} \quad (n \to \infty) \]

which is well supported by the 3 following graphics.
fig.37) Plot of $A(n)n^{0.3970037789}$ taking $A_G(n) = n^{-5}$

fig.38) Plot of $\frac{A(n)}{\log n}n^{0.3970037789}$ taking $A_G(n) = n^{-0.3970037789}$

fig.39) Plot of $A(n)n^{0.1}$ taking $A_G(n) = n^{-0.1}$

There is smooth convergence.
In fact from the tauberian symmetry conjecture for \textit{BHF} and the theorem 3 in part I it is expected that we have in the latter case the more precise formula

$$A(n) = \left(-\frac{1}{0.1g^*(0.1)}\right)n^{0.1} + O\left(n^{-0.3970037789}\right)$$

as shown below.

This graphic should even converge.

18.3 On the complex zeros of \(g^*\)

It is interesting to note that \(g\) satisfies the \textit{HLR} criterion but that it is not the case of the similar function \(g_1\) having 2 discontinuities which is defined by

- \(g_1(x) = \Phi(x)\) if \(x > \frac{1}{3}\)
- \(g_1(x) = x + \frac{1}{3}\) if \(0 < x \leq \frac{1}{3}\)

which clearly doesn’t satisfy the \textit{HLR} criterion. Furthermore the complex zeros of

$$g_1^*(z) = \frac{3^{-1+z}(1-2z)}{z(1-z)} + \frac{2z - 2.3^{-1+z}}{1-z} + \frac{1 - 2^{-1+z}}{(1-z)}$$

are asymptotically located on the line \(x = 1\) and the real part of zeros oscillate around this line as shown with the graphic below where some zeros of \(g_1^*\) are plotted.
fig. 41) Some zeros of $g_1^*$

In an other hand although the complex zeros of $g^*$ are asymptotically located on the vertical line $x = 1$ they stay in the half plane $\Re z > 1$. More precisely letting $\rho$ denoting a complex zero of $g^*$ we have

$$\rho = 1 + \frac{t^{-2}}{2 \log 2} + o(t^{-2}) + it \ (t \to \infty)$$

So it could be interesting and probably easy to understand the relationship between the fact that $g$ satisfies the $HLR$ criterion and the quasi distribution of complex zeros on the vertical line $x = 1$ and in $\Re z > 1$.

19 Generalised tauberian equivalence principle

In fact the tauberian equivalence principle for continuous functions and the tauberian equivalence principle for $BHF$ satisfying the $HLR$ criterion are specific cases. This principle holds apparently for any $FGV$ disregarding the $HLR$ criterion.

19.1 An example of $FGV$ not satisfying the $HLR$ criterion

Let us see this with one of my favourite examples

$$g_{\sqrt{2}}(x) = x\sqrt{2} \left[ -\frac{\log x}{\log \sqrt{2}} \right]$$
Although it is a \textit{BHF} which doesn’t satisfy the \textit{HLR} criterion it is clear that it is a \textit{FGV} and I suspect that the index value is $\alpha(g_{\sqrt{2}}) = \frac{1}{2}$. Recalling that we have

$$g_{\sqrt{2}}^*(z) = \frac{1 - \sqrt{2}^{z-1}}{(1 - z)(1 - \sqrt{2}^z)}$$

and despite the fact that $g_{\sqrt{2}}$ doesn’t fit the analytic conjecture I claim that letting $A_{g_{\sqrt{2}}}(n) = n^{-\beta}$ we still have

$$\beta < \frac{1}{2} \Rightarrow A(n) \sim \left(-\frac{1}{\beta g_{\sqrt{2}}^*(\beta)}\right) n^{-\beta} \ (n \to \infty)$$

The following graphic taking $\beta = \frac{1}{4}$ supports the claim.

fig.42) Plot of $A(n)n^{0.25}$ vs $-\frac{1}{0.25g_{\sqrt{2}}^*(0.25)} = 1.18...$ when $A_{g_{\sqrt{2}}}(n) = n^{-0.25}$.

The attractiveness of the red line is clear as $n \to \infty$. This led me to formulate the generalised tauberian equivalence principle for \textit{FGV} relating the behaviour of discrete sums to integrals in a very general manner since I guess that almost all bounded and measurable functions are \textit{FGV}.
19.2 Conjecture

Let $g$ be a $FGV$ of index $\alpha(g) \in \mathbb{R}$ and consider the discrete-continuous equation

$$
\int_0^x f(t) g\left(\frac{t}{x}\right) \, dt = \sum_{1 \leq k \leq x} a_k g\left(\frac{k}{x}\right) = x^{-\beta}
$$

Then for any $\beta < \alpha(g)$ we have

$$
\int_0^x f(t) dt \sim \sum_{1 \leq k \leq x} a_k \sim \left(\frac{-1}{\beta g^*(\beta)}\right) x^{-\beta} \quad (x \to \infty)
$$

20 A comparison conjecture

The conjectures presented in this note underline that $BHF$ satisfying the $HLR$ criterion have common tauberian properties allowing us to unify the $GRH$ in the realm of tauberian theory. The $HLR$ criterion becomes clearly the bridge between real/complex analysis and discrete mathematics and the corner stone of good variation theory needing to be better understood.

20.1 Conjecture

I recall also the comparison conjecture formulated in [Clo1] where it is proposed to tackle $RH$ by comparing the good variation index of a $BHF g_1$ satisfying the $HLR$ criterion and a $BHF g_2$ which doesn’t satisfy the $HLR$ criterion. Namely it is conjectured that we have in such a situation

$$
\alpha(g_1) \geq \alpha(g_2)
$$

Since $\Phi$ is a $BHF$ satisfying clearly the $HLR$ criterion and $g_{\sqrt{2}}(x) = x \sqrt{\frac{-\log \frac{x}{\sqrt{2}}}{2}}$ appears to be a $BHF$ of index $\frac{1}{2}$ which doesn’t satisfy the $HLR$ criterion we would have

$$
\alpha(\Phi) \geq \alpha(g_{\sqrt{2}}) = \frac{1}{2}
$$

so necessarily $\alpha(\Phi) = \frac{1}{2}$ because we have trivially $\alpha(\Phi) \leq \frac{1}{2}$. Let us see how this conjecture works with another family of functions.

20.2 $BHF$ fitting the analytic conjecture but not the $HLR$ criterion

Consider the family of parametrised generalised Ingham functions
Then we have

\[ g^*(z) = \frac{\zeta(1 - z)\zeta(1 + \lambda - z)\left(1 - 2z^{-\lambda}\right)}{1 - z} \]

so that under \( RH \) and if \( \lambda \notin \{0; \frac{1}{2}\} \) the zeros of \( g^* \) are located on the 3 distinct vertical lines

- \( x = \frac{1}{2}, x = \frac{1}{2} + \lambda \) and \( x = \lambda \).

It is then very interesting to consider these functions for two reasons:

1. If \( \lambda \geq \frac{1}{2} \) \( g \) satisfies clearly the \( HLR \) criterion and so the analytic conjecture works yielding \( \alpha(g) = \frac{1}{2} \) which is supported by experiments.

2. If \( 0 < \lambda < \frac{1}{2} \) it seems that \( g \) doesn’t satisfy \( HLR \) criterion but the analytic conjecture works because \( \alpha(g) = \lambda \) looks plausible from experiments.

So that we can say that for any \( \lambda \geq 0 \) the function \( g \) is a \( FGV \) of index

\[ \alpha(g) = \min \left(\lambda, \frac{1}{2}\right) \]

Therefore assuming that \( RH \) is true the comparison conjecture 20.1 works since when \( 0 < \lambda < \frac{1}{2} \) \( g \) doesn’t satisfy \( HLR \) criterion and indeed \( \alpha(\Phi) = \frac{1}{2} \geq \min (\lambda, \frac{1}{2}) = \lambda \).

In the APPENDIX 2 experimental support is provided.

**20.2.1 Remark**

These functions show that the condition “\( g^* \) satisfies a riemannian functional equation” is essential for conjectures like the conjecture 13.2.

## 21 A striking application of good variation (conjectural) theory

In 1963 using exponential sums and a method of Vinogradov Walfisz \[ Wal \] obtained the following formula

\[
\sum_{k=1}^{n} \frac{\varphi(k)}{k} = \frac{n}{\zeta(2)} + O((\log n)^{2/3}(\log \log n)^{4/3})
\]

and nothing better was ever found regarding the error term. As I know, there is also no improvement of the error term assuming that \( RH \) is true. In the \( \Omega \) direction Montgomery showed in 1987 \[ Mon \] that we have
Let us see how good variation theory relates this problem to the symmetry conjecture 12.1 and so to RH. First it is easy to see that we have
\[
\sum_{k=1}^{n} \frac{\varphi(k)}{k} \Phi \left( \frac{k}{n} \right) = \frac{n + 1}{2}
\]
whence we get
\[
\sum_{k=1}^{n} \left( 2 \frac{\varphi(k)}{k} - \frac{w(k)}{k} \right) \Phi \left( \frac{k}{n} \right) = n
\]
where \( w \) is the sequence introduced in the conjecture 11.2. Hence from the conjecture 12.1 and the case \( \beta = -1 < -\frac{1}{2} \) we get
\[
\sum_{k=1}^{n} \left( 2 \frac{\varphi(k)}{k} - \frac{w(k)}{k} \right) = \frac{-1}{(-1)^{\Phi^{*}(-1)}} n + O(\log(n)^{\varepsilon})
\]
Now from 11.2.1 we have
\[
\sum_{k=1}^{n} \frac{w(k)}{k} = o(1)
\]
and since \( \frac{-1}{(-1)^{\Phi^{*}(-1)}} = \frac{2}{\zeta(2)} \) we get
\[
\sum_{k=1}^{n} \frac{\varphi(k)}{k} = \frac{n}{\zeta(2)} + O(\log(n)^{\varepsilon})
\]
which would be sharper than Walfisz formula.

The following graphic using \( 10^8 \) terms which looks bounded in this range supports the \( O(\log(n)^{\varepsilon}) \).
22 Concluding remarks

It becomes quite clear that RH is both an arithmetical problem (HLR criterion) and an analytic problem (analytic conjecture) confirming the subtle duality already mentioned in the introduction of [Clo1]. It is also an inverse problem [Tar] since the question would not be

- Where are nontrivial zeros located?

which is in fact the answer to the problem, not the question, but rather

- Does the Ingham function satisfy the HLR criterion?

which should be the proper question and the inverse problem nature of RH was already considered in a different field [Lap].

Last but not least it seems possible to extend the set of FGV to functions satisfying \( \lim_{x \to 0} g(x) = \infty \) with for instance \( g(x) = O(\log x) \) as \( x \to 0 \). A simple example is given by

\[
g(x) = x \sum_{k \geq 1} \left\lfloor \frac{1}{kx} \right\rfloor
\]

since we have as \( x \to 0 \)

\[
g(x) = -\log x + 2\gamma - 1 + o(1)
\]

Next \( g^*(z) = \frac{(1-z)^2}{1-z} \) and I claim that \( g \) fits the analytic conjecture for BHF and the tauberian symmetry conjecture with an additional log function as a factor since \( g^* \) has double nontrivial zeros. See APPENDIX 3 for experimental support.
References


[Bor2] Peter Borwein et al, Signs change in sums of the Liouville function, Mathematics of computation (2008)


[Clo1] B. Cloitre, Broken harmonic functions (2014)

[Clo2] B. Cloitre, Mock Liouville functions (2014)


[Clo4] B. Cloitre, Good variation (2011)


(gp) PARI/GP computer algebra system
APPENDIX 1

The tauberian equivalence principle for polynomials of degree 1

Let

• \( g(x) = (1 - \lambda)x + \lambda \) where \( \lambda \in ]0, 1[ \).
• \( g^*(z) = \frac{1 - \lambda}{1-z} - \frac{\lambda}{z} = \frac{\lambda - \lambda}{z(1-z)} \)

Suppose that we have for some \( \beta \in \mathbb{R} \) and any \( x > 0 \)

\[
\int_0^x f(t)g \left( \frac{t}{x} \right) dt = \sum_{0 < k \leq x} a(k)g \left( \frac{k}{x} \right) = x^{-\beta}
\]

I shall prove the following proposition

\[
0 < \beta < \lambda \Rightarrow \int_0^x f(t)dt \sim \sum_{0 < k \leq x} a(k) \sim \left( \frac{1}{\beta g^*(\beta)} \right) x^{-\beta} \quad (x \to \infty)
\]

Proof

The continuous case

Multiplying the equation \( \int_0^x f(t)g \left( \frac{t}{x} \right) dt = x^{-\beta} \) by \( x \) we get

\[
(1 - \lambda) \int_0^x tf(t)dt + \lambda x \int_0^x f(t)dt = x^{-\beta+1}
\]

since all is smooth we can differentiate (1) with respect to \( x \) and we get letting \( y = \int_0^x f(t)dt \)

\[
xy' + \lambda y = (1 - \beta)x^{-\beta}
\]

and the solution of (7) is given by

\[
\int_0^x f(t)dt = Kx^{-\lambda} + \frac{\beta - 1}{\beta - \lambda}x^{-\beta}
\]

so that we have

\[
0 < \beta < \lambda \Rightarrow \int_0^x f(t)dt \sim \left( \frac{\beta - 1}{\beta - \lambda} \right) x^{-\beta} \quad (x \to \infty)
\]

and since \( \beta g^*(\beta) = \frac{\lambda - \beta}{1 - \beta} \) the proposition is proved for the continuous case.
The discrete case

Multiplying the equation \( \sum_{0 \leq k \leq n} a(k)g\left( \frac{k}{n} \right) = n^{-\beta} \) by \( n \) we get

\[
(1 - \lambda) \sum_{0 \leq k \leq n} ka(k) + \lambda n \sum_{0 \leq k \leq n} a(k) = n^{-\beta + 1}
\] (8)

next writing \( A(n) = \sum_{0 \leq k \leq n} a(k) \) and using the difference operator
\( \Delta (x(n)) = x(n) - x(n-1) \) on (8) we get the difference equation analogue to
the Euler-Cauchy differential equation (7)

\[
n\Delta (A(n)) + \lambda A(n-1) = (n^{-\beta+1} - (n-1)^{-\beta+1})
\] (9)

Then letting \( h(n) = n^{-1} (n^{-\beta+1} - (n-1)^{-\beta+1}) \) the solution of (9) is given
explicitely for \( n \geq 2 \) by

\[
A(n) = h(n) + \frac{(1 - \lambda)}{n!} \left( \frac{A(1)}{1 - \lambda} + \sum_{k=2}^{n-1} h(k) \frac{k!}{(1 - \lambda)_k} \right)
\] (10)

Now it is easy to see that we have the 3 asymptotic formulas as \( n \to \infty \):

\[
h(n) = (1 - \beta) n^{-\beta-1} + O(n^{-\beta-2})
\]

\[
\frac{(1 - \lambda)n}{n!} = \frac{1}{\Gamma(1 - \lambda)} \left( n^{-\lambda} + O(n^{-\lambda-1}) \right)
\]

\[
\frac{k!}{(1 - \lambda)_k} = \Gamma(1 - \lambda) \left( k^\lambda + O(k^{\lambda-1}) \right)
\]
yielding

\[
h(k) \frac{k!}{(1 - \lambda)_k} = \Gamma(1 - \lambda) \left( (1 - \beta) k^{\lambda-\beta -1} + O(k^{\lambda-\beta-2}) \right)
\]

whence (10) becomes for \( 0 < \beta < \lambda \)

\[
A(n) = (1 - \beta) n^{-\beta-1} + O(n^{-\beta-2}) + (n^{-\lambda} + O(n^{-\lambda-1})) \left( (1 - \beta) \sum_{k=2}^{n-1} (k^{\lambda-\beta -1} + O(k^{\lambda-\beta-2})) + O(1) \right)
\] (11)

hence since \( \lambda - \beta - 1 > -1 \) the equation (11) reduces to

\[
A(n) = \frac{1 - \beta}{\lambda - \beta} n^{-\beta} + O(n^{-\lambda})
\]

which proves the proposition for the discrete case.
APPENDIX 2

Here I consider the family of parametrised generalised Ingham functions

\[ g(x) = x \sum_{k \geq 1} \frac{(-1)^{k-1}}{k^\lambda} \left\lfloor \frac{1}{kx} \right\rfloor \]

(more material to be added)
Case $\lambda \geq \frac{1}{2}$

The $HLR$ criterion is satisfied

fig.44) Plot of $na(n)$ when $A_g(n) = 2^{-n}$ and $\lambda = \frac{1}{2}$

fig.45) Plot of $na(n)$ when $A_g(n) = 2^{-n}$ and $\lambda = \frac{3}{4}$
The analytic conjecture works

fig.46) Plot of $A(n) n^{1/2}$ when $A_g(n) = 2^{-n}$ and $\lambda = \frac{1}{2}$

fig.47) Plot of $A(n) n^{1/2}$ when $A_g(n) = 2^{-n}$ and $\lambda = \frac{3}{4}$
Case $0 \leq \lambda < \frac{1}{2}$

The $HLR$ criterion is not satisfied

fig.48) Plot of $na(n)$ when $A_g(n) = 2^{-n}$ and $\lambda = \frac{1}{3}$

fig.49) Plot of $na(n)$ when $A_g(n) = 2^{-n}$ and $\lambda = 0$
However the analytic conjecture works

fig.50) Plot of $A(n)n^{1/3}$ when $A_g(n) = 2^{-n}$ and $\lambda = \frac{1}{3}$

fig.51) Plot of $A(n)$ when $A_g(n) = 2^{-n}$ and $\lambda = 0$
fig.56) Plot of $A(n)n^{1/2}(\log n)^{-1}$ when $A_g(n) = 2^{-n}$

It looks bounded confirming that $g$ should fit the analytic conjecture and the tauberian symmetry conjecture providing we multiply the remainder term by $\log(n)$ in each case.