The space of functions of good variation An algebraic topological approach to RH

Benoit Cloitre

April 14, 2017

Abstract

In this article I try to developp good variation theory in the realm of Algebraic Topology as suggested in [Cl1]. Indeed I feel that solving the Riemann hypothesis and its generalisations using Good Variation Theory (GVT) could require a final step made of topological arguments. To this end I introduce the topological space of functions of good variation (FGV) and suitable subspaces associated to good variation invariants and continuous morphisms. Then I make a bunch of conjectures allowing me to transform RH into a topological property.

Introduction

Good variation theory aims to understand better the Riemann Hypothesis from a Tauberian view point and I gave an exposition of this theory in [Cl1]. In the concluding remarks I wrote:

"*To prove the anti HLR conjecture however, I believe that an algebraic approach could be promising. The idea consists in considering the space of FGV and the subspace of affine functions by parts on* $]0,1]$ *and to look for Tauberian invariants.*"

In this article I work out what I had in mind and state some conjectures allowing me to derive that RH could be a topological property. For instance, from a good variation view point, the Ingham function $\Phi(x) = x\left\lfloor \frac{1}{x} \right\rfloor$ would be homotopic to the much simpler function $\Phi_0(x) = \frac{x+1}{2}$.

To achieve this aim I give in section 1 a new existence theorem for FGV which is much broader that the theorem 1.1 in [Cl1] since all Riemann integrable functions are FGV.

In section 2 I introduce a classification of FGV underlying that GVT can't rely on analysis only and different spaces. Subspaces of FGV are considered in section 3 where I state the fundamental index conjecture (3.6).

Next in section 4 some values of the good variation index are conjectured to be topological invariants for peculiar dense subspaces of FGV. In section 5 I discuss the existence of continuous morphisms in these subspaces of FGV

and I state another fundamental conjecture: the Hardy-Littlewood-Ramanujan criterion is a topoligical invariant.

Notations and definitions are the same as those given in [Cl1] and for the sake of clarity I will recall time to time some of the main definitions throughout the paper.

1 The existence theorem

In [Cl1] I proved that affine functions are FGV and I conjectured that all continuous functions on $[0,1]$ are FGV. To prove the latter conjecture I thought to prove it for polynomials first and then the Weiestrass approximation theorem should work for any continuous function. It turn's out that regarding existence of FGV much more is true and the proof is easier than expected. Indeed recently I succeeded to prove the following existence theorem using simply the least-upper-bound property in $\mathbb{R}$.

This theorem is an existence theorem since it says nothing about the value of the good variation index of a given FGV which is a delicate question as shown hereafter in section 2.

Using a completeness property wich pertains to set theory and analysis is also a good start to illustrate the possible connection between GVT and algebraic topology.

Before stating the theorem I recall the primary definition of FGV.

Function of good variation (primary definition)

A bounded function g defined on $]0,1]$ is a function of good variation (FGV) of index $\alpha(g)$ if letting

$$A_g(n) := \sum_{k=1}^{n} a_k g\left(\frac{k}{n}\right) = n^{-\beta}$$

and $A(n) := \sum_{k=1}^{n} a_k$ we have the 2 following tauberian properties

1. $\beta < \alpha(g) \Rightarrow A(n) \sim C(\beta) n^{-\beta} \ (n \to \infty)$ where $C(\beta) \neq 0$
2. $\beta > \alpha(g) \Rightarrow A(n) n^{\beta}$ is unbounded.

Remark 0 The primary definition could be refined. For instance one could add a remainder term i.e. $A_g(n) = n^{-\beta} + f(n)$ where $f(n)$ and $|a_n|$ satisfy suitable growth conditions (see for instance [Cl2], p. 44 where $g = \Phi$ the Ingham function). But it doesn't matter here since this primary definition is the essential one and sufficient for my purpose.

1.1 The existence theorem

Let g be defined on $]0,1]$ and Riemann integrable on $[0,1]$. Then g is a FGV according to the primary definition of FGV.

Proof of theorem 1.1

If $\beta < 0$ and $|\beta|$ is large enough and since g is Riemann integrable it can be shown (details are omitted) that we have:

$$A_g(n) = n^{-\beta} \Rightarrow A(n) \sim -\frac{1}{\beta g^{\star}(\beta)} n^{-\beta} \ \ (n \to \infty) \tag{1}$$

where for $\Re z < 0$ $g^{\star}(z) := \int_0^1 g(t)t^{-z-1}dt$ is the little Mellin transform of g. Next let $P(\beta) = 1$ $(= 0)$ denotes the fact that the property (1) holds (doesn't hold) and define the set

$$B_1 := \{\beta \in \mathbb{R} \ : \ P(\beta) = 1\}$$

The set B_1 in non empty from (1) so that from the least-upper-bound property in $\mathbb{R}$ there is a unique value $\alpha(g) = \max\{\beta \in B_1\}$ such that we have:

$$\beta < \alpha(g) \Rightarrow P(\beta) = 1 \ \wedge \ \beta > \alpha(g) \Rightarrow P(\beta) = 0$$

which is precisely the definition of the good variation index of g.

□

2 Types and examples of FGV

In order to classify FGV I will make use of the value of the good variation index but I need also to introduce the analytic index which is defined by:

$$\eta(g) := \min\{\Re(\rho) \ | \ g^{\star}(\rho) = 0\}$$

Often we have $\alpha(g) = \eta(g)$ but it is not always the case. That's why it is interesting to make a categorization of FGV depending on these values.

2.1 Types of FGV

There are different kind of FGV and I distinguish three types of FGV : Algebraic, Analytic or Degenerate, depending on the relationship between the good variation index $\alpha(g)$ and the analytic index $\eta(g)$. Letting g denotes a FGV these three types are defined as follows:

1. g is Algebraic if $\alpha(g)$ exists and is finite but $\eta(g)$ doesn't exist.
2. g is Analytic if $\eta(g)$ exists and is finite and $\alpha(g) = \eta(g)$.
3. g is Degenerate if $\eta(g)$ exists and is finite and $\alpha(g) \neq \eta(g)$.

This classification underlines that my Tauberian approach to RH is not easy to handle and since GVT is not a purely analytic theory.

For instance in GVT the statement $\alpha(\Phi) = \eta(\Phi) = \frac{1}{2}$ where $\Phi(x) = x\left\lfloor \frac{1}{x} \right\rfloor$ is the Ingham function implies that RH is true and so Φ would be an Analytic FGV.

But Degenerate FGV exist (as we shall see) so that the Ingham function Φ could be a Degenerate one after all. Hopefully I don't think so because Φ has two peculiar properties which are of paramount importance in my theory:

- Φ satisfies the Hardy-Littlewood-Ramanujan criterion
- $\Phi^\star(z) = \frac{\zeta(1-z)}{1-z}$ satisfies a Riemann functional equation

These properties will be discussed in more details later. Hereafter I provide examples illustrating the different types of FGV. These examples are trivial or nontrivial, proved or conjectured. If there is a proof this one is omitted and if it is a conjecture it is always on the basis of many clear experiments (also omitted here).

2.2 Algebraic FGV

Let $\lambda \in]0,1[$ and let g be defined as follows:

- $g(x) = 1$ if $x \in [0, \frac{1}{2}[\cup]\frac{1}{2}, 1]$
- $g\left(\frac{1}{2}\right) = \lambda$

Then we have $g^\star(z) = -\frac{1}{z}$ so that $\eta(g)$ doesn't exist but I proved that $\alpha(g) = \frac{\log(1-\lambda)}{\log 2}$ exists and is finite. Therefore g is an Algebraic FGV for any $\lambda \in]0,1[$.

Remark 1 I think that any fonction g almost constant on $[0,1]$ with finitely many discontinuities at rational values is a Degenerate FGV.

2.3 Analytic FGV

2.3.1 Trivial examples

Let $\lambda \in]0,1[$ and g be defined as follows:

- $g(x) = (1-\lambda)x + \lambda$

Then we have $g^\star(z) = \frac{(1-\lambda)}{1-z} - \frac{\lambda}{z}$ so that $\eta(g) = \lambda$ and I proved (theorem 1.1 in [Cl1]) that g is a FGV satisfying $\alpha(g) = \lambda$. Hence g is an Analytic FGV.

2.3.2 Non trivial examples

The Ingham function The Ingham function $\Phi(x) = x\left\lfloor \frac{1}{x} \right\rfloor$ satisfies $\Phi^\star(z) = \frac{\zeta(1-z)}{1-z}$. Then assuming the RH and the anti HLR conjecture in [Cl1] we have $\alpha(\Phi) = \eta(\Phi) = \frac{1}{2}$ so that the Ingham function would be an Analytic FGV.

Other broken harmonic functions Let $\lambda > 1$ and $g(x) = \lambda^{\left\lfloor -\frac{\log x}{\log \lambda} \right\rfloor}$ which satisfies $g^{\star}(z) = \frac{1-\lambda^{1-z}}{(1-z)(1-\lambda^{z})}$ so that $\eta(g) = 1$. Then for any integer value $\lambda \geq 2$, I proved that $\alpha(g) = 1$ hence in this case g is an Analytic FGV.

A smooth function of unbounded variation Let $g(x) = 1 + \sin(\log x)$ then we have $g^{\star}(z) = -\frac{1}{z} - \frac{1}{1+z^2}$ hence $\eta(g) = -\frac{1}{2}$ and it does appear that $\alpha(g) = -\frac{1}{2}$ so that g would be an Analytic FGV.

2.4 Degenerate FGV

2.4.1 Trivial examples

Let $g(x) = x + x^2$ then we have $g^{\star}(z) = \frac{1}{1-z} + \frac{1}{2-z}$ so that $\eta(g) = \frac{3}{2}$ and it can be shown that we have $\alpha(g) = 0$. Therefore g is a Degenerate FGV.

Remark 2 In fact many FGV of index zero are Degenerate FGV. Indeed for any Analytic FGV g satisfying $\alpha(g) = \eta(g) \neq -1$ the function $h(x) = xg(x)$ has index $\alpha(h) = 0$ whereas $\eta(h) = \eta(g) + 1 \neq 0$.

2.4.2 Non trivial examples

The two following examples are very interesting and conjectured to be Degenerate *FGV*.

The function $g(x) = \sqrt{2}^{\left\lfloor -\frac{\log x}{\log \sqrt{2}} \right\rfloor}$ satisfies $g^{\star}(z) = \frac{1-\sqrt{2}^{1-z}}{(1-z)\left(1-\sqrt{2}^{z}\right)}$ so that $\eta(g) = 1$ but I conjectured that $\alpha(g) = \frac{1}{2}$ which is very well supported by experiments. Therefore g would be a nontrivial Degenerate FGV.

Let $\xi = \frac{-2+\sqrt{10-2\sqrt{5}}}{\sqrt{5}-1} = 0.284079...$then Davenport and Heilbronn considered the analytic continuation of the Dirichlet series $H(s) = \sum_{n\geq 1} \frac{h(n)}{n^s}$ where h is the 5-periodic sequence $1, \xi, -\xi, -1, 0, ...$ and proved that it has nontrivial zeros off the critical line despite the fact that H satisfies a Riemann functional equation. Moreover the real parts of these zeros are dense in $[0, 1]$. Then considering $g_H(x) := x \sum_{1\leq k\leq \left\lfloor \frac{1}{x} \right\rfloor} h(k) \left\lfloor \frac{1}{kx} \right\rfloor$ we have $g_H^{\star}(z) = \frac{\zeta(1-z)H(1-z)}{1-z}$ so that $\eta(g_H) = 0$ but experiments suggest that $\alpha(g_H) = \frac{1}{2}$. Therefore g_H would be a nontrivial Degenerate FGV.

3 The space of FGV and some subspaces

From what we see above it is natural to consider the space of FGV as a global structure and then to exhibit special subspaces with peculiar properties related to good variation.

3.1 Riemann integrable functions as a vector space of FGV

Theorem 3.1

Let E_R denotes the set of bounded Riemann integrable functions on]0,1]. Then, assuming that the function $g = 0$ is a FGV, E_R is a real vector space of FGV.

Proof

It is clear that if g_1 and g_2 are bounded Riemann integrable so does $\lambda g_1 + \mu g_2$ where $(\lambda, \mu) \in \mathbb{R}^2$ and from the existence theorem 1.1 we can say that $g_1, g_2, \lambda g_1 + \mu g_2$ are FGV. Therefore E_R is a real vector space of FGV.

Remark 3 We have many choices for a norm such as $\|g\|_\infty = \sup\{|g(t)| \ \ , 0 < t \leq 1\}$.

Remark 4 I think that E_R contains all FGV and probably one can show that a function which is not Riemann integrable can't be a FGV.

3.2 The space E_A

In fact the vector space structure won't be necessary since the structure of topological space will suffice for my purpose. Let us define E_A as the space of affine functions by parts on $]0, 1]$, positive and continuous on the left, increasing by parts and satisfying $g(x) = O(1)$ as $x \to 0$. From now on I will consider mainly the set E_A which is a subset of E_R and which has the following interesting closure property

$$(g_1, g_2) \in E_A \times E_A \wedge (\lambda_1, \lambda_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \Rightarrow \lambda_1 g_1 + \lambda_2 g_2 \in E_A$$

3.2.1 Caracterisation of E_A

It is convenient to see that $g \in E_A$ if and only if there are $N \in \mathbb{Z}_{\geq 1}$ (which can be infinite) and three real sequences $(I_i)_{1 \leq i \leq N}$, $(u_i)_{1 \leq i \leq N}$, $(v_i)_{1 \leq i \leq N}$ satisfying

- $0 = I_N < I_{N-1} < .... < I_1 = 1$
- $\forall i \in \{1, 2, ..., N\}$ we have $u_i \geq 0$ and $0 \leq u_i I_i + v_i \ll 1$
- $I_i < x \leq I_{i+1} \Rightarrow g(x) = u_i x + v_i$

The following functions already introduced in [Cl1] are special functions in E_A and are particularly important for my good variation approach to RH.

3.2.2 Broken Harmonic Function

If $N = \infty$ then g is a so called Broken Harmonic Function (BHF). For instance

- The Ingham function $\Phi(x) = x \left\lfloor \frac{1}{x} \right\rfloor$ for which $I_i = \frac{1}{i}, u_i = i, v_i = 0$.
- For $\lambda > 1$, $g(x) = x \lambda^{\left\lfloor -\frac{\log x}{\log \lambda} \right\rfloor}$ for which $I_i = \frac{1}{\lambda^{i-1}}, u_i = \lambda^{i-1}, v_i = 0$.

3.2.3 Little Mellin transform in E_A

It is also convenient to see that the little Mellin transform of $g \in E_A$ is given by (keeping previous notations)

- $g^\star(z) = \int_0^1 g(t)t^{-z-1}dt = \frac{1}{1-z}\sum_{n=1}^N u_n\left(I_n^{1-z} - I_{n+1}^{1-z}\right) - \frac{1}{z}\sum_{n=1}^N v_n\left(I_n^{-z} - I_{n+1}^{-z}\right)$

3.3 The space E_{HLR}

In [Cl1] the Hardy-Littlewood-Ramanujan (HLR) criterion plays a crucial role. I recall it. A function g satisfies the HLR criterion if for any $\beta \geq 0$ we have the property

$$A_g(n) = n^{-\beta} \Rightarrow \forall\varepsilon > 0 \lim_{n\to\infty} a(n)n^{1-\varepsilon} = 0$$

We then say that g is HLR. So it is natural to consider the subspace E_{HLR} of E_R consisting of FGV which are HLR.

3.4 The space E_{TFE}

A function g satisfies the TFE criterion if the little Mellin transform $g^\star$ satisfies a "Trivial" functional equation i.e. $(1-z)g^\star(z)zg^\star(1-z)$ has no zero in the critical strip $0 < \Re z < 1$. In the sequel E_{TFE} denotes the subspace of E_R such that $(1-z)g^\star(z)$ satisfies a trivial functional equation.

3.5 The space E_{RFE}

A function g satisfies the RFE criterion if the little Mellin transform $g^\star$ satisfies a Riemann functional equation i.e. $\frac{(1-z)g^\star(z)}{zg^\star(1-z)}$ has no zero in the critical strip $0 < \Re z < 1$. From now on E_{RFE} denotes the subspace of E_R such that $(1-z)g^\star(z)$ satisfies a Riemann functional equation.

Before discussing some invariants it seems interesting to state the index conjecture in E_A.

3.6 The fundamental index conjecture in E_A

It is clear that when g is C^0 on $[0,1]$ then $\alpha(g) = \eta(g)$. When there is a discontinuity at zero things are not so simple. Nevertheless thanks to the HLR criterion I claim that we have

$$g \in E_A \cap E_{HLR} \Rightarrow \alpha(g) = \eta(g)$$

but the converse doesn't hold (in general). For instance let $g(x) = x2^{\left\lfloor \frac{\log x}{\log 2} \right\rfloor} + x4^{\left\lfloor \frac{\log x}{\log 4} \right\rfloor}$ then we have $g^\star(z) =$ and experiments show clearly that g is not HLR but it does appear that $\alpha(g) = \eta(g) = \frac{1}{2}$.

4 Topological Invariants

It took me a long time before realising that the index of good variation can caracterise dense sets of FGV and could have an algebraic nature.

4.1 The log invariance property

This property is common to all FGV and was the first one I observed historically. More precisely suppose that g is a FGV of index $\alpha(g)$. Suppose also that $L(n)$ is the sharpest slowly varying function such that

$$A_g(n) = n^{-\alpha(g)} \Rightarrow A(n) \ll n^{-\alpha(g)} L(n)$$

then we have the invariance sharp asymptotic property

$$A_g(n) = n^{1-\alpha(g)} \Rightarrow A(n) = \frac{n^{1-\alpha(g)}}{(1-\alpha(g))g^\star(\alpha(g)-1)} + O\left(n^{-\alpha(g)} L(n)\log(n)\right)$$

Although this "analytic" property shed few light on the value of $\alpha(g)$, it underlines that the good variation index should have invariance properties. It is then more interesting to consider the value of $\alpha(g)$ and to look for properties of g which can caracterise dense sets of functions. It appears that $0, 1/2, 1$ are 3 values of $\alpha(g)$ which can caracterise dense subspaces of E_R.

4.2 Zero as an invariant

Zero is a topological invariant in many subspaces of E_R. Indeed there are many FGV such that $\alpha(g) = 0$ and we can consider spaces of FGV with this property dense enough so that continuous morphisms preserve the property. For instance, let g be positive, increasing and continuous on $[0,1]$ with $g(0) = 0$ and $g(1) > 0$ then g is a FGV of index zero. More generally I claim that if:

- $g \in E_R$
- $\lim_{x\to 0} g(x) = \lim_{z\to 0} zg^\star(z) = 0$

then g is a FGV of good variation index zero.

In other words 0 is for instance a topological invariant in xE_A where $xE_A = \{h\ , h(x) = xg(x)\ , g \in E_A\}$.

4.3 One as an invariant

Let $g_\lambda(x) = x\lambda^{\lfloor -\frac{\log x}{\log \lambda}\rfloor}$ which satisfies $g_\lambda^\star(z) = \frac{1-\lambda^{1-z}}{(1-z)(1-\lambda^z)}$. We see in 2.3.2 that we have $\eta(g_\lambda) = \alpha(g_\lambda) = 1$ if and only if $\lambda \geq 2$ is an integer value. Now observe that we have $g_\lambda^\star(z)g_\lambda^\star(1-z) = \frac{1}{z(1-z)}$ therefore the function $h(z) = (1-z)g_\lambda^\star(z)$ satisfies the trivial functional equation $h(z)h(1-z) = 1$. Hence if $\lambda \geq 2$ is an integer value we have $g_\lambda \in E_A \cap E_{HLR} \cap E_{TFE}$ and $\alpha(g_\lambda) = \eta(g_\lambda) = 1$.

This led me to claim that we have the property

$$g \in E_A \cap E_{HLR} \cap E_{TFE} \wedge g(0) > 0 \Rightarrow \alpha(g) = \eta(g) = 1$$

and the set $E_A \cap E_{HLR} \cap E_{TFE}$ is dense in E_R.

In other words 1 is a topological invariant in $E_A \cap E_{HLR} \cap E_{TFE}$.

The trivial example

The function $f(x) = 1$ can be considered as the limit case $\mu \to 1$ of $f_\mu(x) = (1-\mu)x + \mu$ which is a FGV of index $\alpha(f_\mu) = \eta(f_\mu) = \mu$. So we have formally $\alpha(f) = \lim_{\mu \to 1} \alpha(f_\mu) = 1$ and $f \in E_A \cap E_{HLR} \cap E_{TFE}$ since $f^\star(z) = -\frac{1}{z}$ satisfies $f^\star(z) f^\star(1-z) = \frac{1}{z(1-z)}$.

4.4 One half as an invariant

Consider the Selberg class S. For each $F \in S - \{\zeta\}$ let $F(s) = \sum_{n \geq 1} \frac{f(n)}{n^s}$ and $g(x) = x \sum_{1 \leq k \leq x^{-1}} f(k) \left\lfloor \frac{1}{kx} \right\rfloor$. Then g is HLR from the theorem 5.1 in [Cl1] and $(1-z)g^\star(z) = \zeta(1-z)F(1-z)$ satisfies a Riemann functional equation. From the anti HLR conjecture (§3 in [Cl1]) we must have $\alpha(g) = \frac{1}{2}$. Let us recall this conjecture hereafter.

The anti HLR conjecture for BHF Let g be a BHF such that:

- $\lim_{x \to 0} g(x) \neq 0$ exists
- $(1-z)g^\star(z)$ satisfies a Riemann functional equation

Then if $g^\star$ has a zero in the half-plane $\Re z < \frac{1}{2}$ g is not HLR.

So we see that one half is an invariant for a large class of functions. However I don't think that the set of FGV generated by the Selberg class is dense enough and to me many more functions in E_A share the property. This led me to claim that we have the property

$$g \in E_A \cap E_{HLR} \cap E_{RFE} \wedge g(0) \neq 0 \Rightarrow \alpha(g) = \eta(g) = \frac{1}{2}$$

and I claim that the set $E_A \cap E_{HLR} \cap E_{RFE}$ is dense in E_R.

In other words $\frac{1}{2}$ is a topological invariant in $E_A \cap E_{HLR} \cap E_{RFE}$.

The simplest example

Let $\Phi_0(x) = \frac{x+1}{2}$ which is a FGV of index $\alpha(\Phi_0) = \frac{1}{2}$ (see theorem 1 [Cl1]) and is in $E_A \cap E_{HLR} \cap E_{RFE}$. Indeed Φ_0 is affine by part and increasing, satisfies the HLR criterion and we have $\Phi_0^\star(z) = \frac{1}{2}\left(\frac{1}{1-z} - \frac{1}{z}\right)$ so that $h(z) = (1-z)\Phi_0^\star(z)$ satisfies the Riemann functional equation $h(z) = \left(\frac{z-1}{z}\right) h(1-z)$.

Remark 5 In view of the example related to the Davenport and Heilbronn zeta function given in 2.4.2 one may suspect that a Riemann functional equation is sufficient for the invariance of the good variation index in $E_{HLR} \cap E_{RFE}$ but not for the invariance of the analytic index. Hence we should have

$$g \in E_A \cap E_{RFE} \wedge g(0) > 0 \Rightarrow \alpha(g) = \frac{1}{2}$$

5 Morphisms

Here I show that there are continuous morphisms in E_A preserving the HLR criterion and I claim that there are continuous morphisms in E_A preserving both the HLR criterion and a functional equation property (TFE or RFE).

5.1 Continuous morphism in $E_A \cap E_{HLR}$

It is easy to see that continuous morphisms in E_A preserving the HLR criterion exist. The linear morphism is a concrete example.

The linear morphism preserves the HLR criterion

From now on $E_A^{'}$ denotes the subspace of E_A of functions g satisfying $\lim_{x\to 0} g(x) = \ell > 0$ exists.

Let $(g_1, g_2) \in E_A^{'} \cap E_{HLR}$ and for $x \in [0,1]$ let us consider the linear morphism

$$T_x(g_1, g_2) = xg_1 + (1-x)g_2$$

then $T_x(g_1, g_2)$ is a continuous morphism in E_A such that

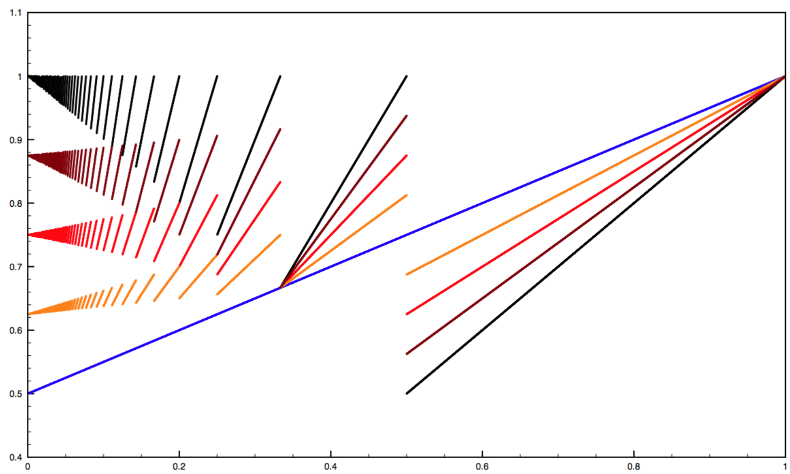
- $T_0(g_1, g_2) = g_2$
- $T_1(g_1, g_2) = g_1$
- $T_x(g_1, g_2)$ is HLR (proof left to the reader).

Therefore it exists continuous morphisms in E_A preserving the HLR criterion.

In the figure 1 below one can see how this linear morphism acts on $(g_1, g_2) = (\Phi, \Phi_0)$ where $\Phi_0(x) = \frac{x+1}{2}$ could be called the trivial Ingham function. However this morphism doesn't preserve the good variation index. Indeed we have under RH $\alpha(\Phi) = \alpha(\Phi_0) = \frac{1}{2}$ but for $0 < x < 1$ it is expected that $\eta(T_x(\Phi, \Phi_0)) \leq 0$ from a theorem of Titchmarsh.

Fig.1) Ingham function trivial smoothing

Plot of $T_x(\Phi, \Phi_0)$ *for* $x = 1, 0.75, 0.5, 0.25, 0$



5.1.1 The HLR invariant conjecture

In fact the previous result led me to state the following fundamental conjecture which is supported by experiments when I take linear combinations of various distinct functions from $E'_A \cap E_{HLR}$.

The HLR invariant conjecture Let

- $g_1 \in E'_A \cap E_{HLR}$ and $g_2 \in E'_A \cap E_{HLR}$

Let $\Psi : [0, 1] \to E'_A$ be a continuous morphism satisfying:

- $\Psi(0) = g_1$
- $\Psi(1) = g_2$

Then we have

$$x \in]0, 1[\Rightarrow \Psi(x) \in E'_A \cap E_{HLR}$$

5.2 Morphisms preserving the HLR criterion and a functional equation

We see that a continuous morphism preserving the HLR criterion doesn't preserve in general the index of good variation. However due to the invariance properties described in 4.2, 4.3 and 4.4 one can expect that suitable morphisms exist preserving the HLR criterion and a functional equation. This led me to state the following conjecture

5.2.1 The morphism conjecture in E_A

Let $g_1, g_2 \in E_A$ which are both HLR and such that $h_1(z) = (1-z)g_1^\star(z)$ and $h_2(z) = (1-z)g_2^\star(z)$ satisfy the same kind of functional equation (TFE or RFE). Then there exists a continuous morphism $T_x(g_1, g_2)$ such that

- $T_0(g_1, g_2) = g_1$
- $T_1(g_1, g_2) = g_2$
- $\forall x\ [0,1]\ T_x(g_1, g_2) \in E_A$
- $\forall x\ [0,1]\ T_x(g_1, g_2)$ is HLR
- $\forall x\ [0,1]\ (1-z)T_x(g_1, g_2)^\star(z)$ satisfies the same functional equation than h_1 and h_2

In particular it exists a continuous morphism $T_x(\Phi, \Phi_0)$ in $E_A \cap E_{HLR} \cap E_{RFE}$ such that we have:

- $T_0(\Phi, \Phi_0) = \Phi$ and $T_1(\Phi, \Phi_0) = \Phi_0$
- $T_x(\Phi, \Phi_0) \in E_A \cap E_{HLR} \cap E_{RFE}$ for any $x \in [0,1]$

5.2.2 Specialising the morphism conjecture

Using notations in 3.2.1 I take the sequence $I_n = \frac{1}{n}$ and I make the stronger claim that for any fixed $i \in \mathbb{Z}_{\geq 1}$ there are two continuous functions of $x \in [0,1]$ $u_i(x)$ and $v_i(x)$ satisfying

- $u_i(0) = i$, $u_i(1) = \frac{1}{2}$
- $v_i(0) = 0$, $v_i(1) = \frac{1}{2}$

and such that the function f_x given for $\Re z < 0$ by

$$f_x(z) = \frac{1}{1-z}\sum_{n=1}^{\infty} u_n(x)\left(\frac{1}{n^{1-z}} - \frac{1}{(n+1)^{1-z}}\right) - \frac{1}{z}\sum_{n=1}^{\infty} v_n(x)\left(\frac{1}{n^{-z}} - \frac{1}{(n+1)^{-z}}\right)$$

is well defined and has a meromorphic continuation satisfying a Riemann functional equation for any $x \in [0,1]$.

Remark 6 So far I didn't succeed to find explicit continuous sequence-functions $u_i(x)$ and $v_i(x)$ but my intuition is that they exist. In other words there must be some FGV in $E_A \cap E_{HLR} \cap E_{RFE}$ "between" Φ and Φ_0 and it would be fine to exhibit a non trivial example in order to support this approach.

5.3 RH as a topological property

I recall that from the conjecture 4.4 we would have

$$\Phi \in E_A \cap E_{HLR} \cap E_{RFE} \wedge \Phi(0) = 1 > 0 \Rightarrow \alpha(\Phi) = \eta(\Phi) = \frac{1}{2}$$

hence RH would be true. Moreover the Ingham function Φ could be continuously deformed into $\Phi_0(x) = \frac{x+1}{2}$ in $E_A \cap E_{HLR} \cap E_{RFE}$. Therefore from a good variation topological view point the Ingham function Φ would be homotopic to the trivial Ingham function Φ_0.

5.4 Variation on the morphism conjecture 5.2.2

With the previous conjecture in mind I relax the conditions in 5.2.2 as follows.

For any fixed $i \in \mathbb{Z}_{\geq 1}$ there are two continuous functions of $x \in [0, 1]$ $u_i(x)$ and $v_i(x)$ satisfying

- $u_i(0) = i$, $u_i(1) = \frac{1}{2}$
- $v_i(0) = 0$, $v_i(1) = \frac{1}{2}$

and such that the function f_x given for $\Re z < 0$ by

$$f_x(z) = \frac{1}{1-z} \sum_{n=1}^{\infty} u_n(x) \left(\frac{1}{n^{1-z}} - \frac{1}{(n+1)^{1-z}} \right) - \frac{1}{z} \sum_{n=1}^{\infty} v_n(x) \left(\frac{1}{n^{-z}} - \frac{1}{(n+1)^{-z}} \right)$$

is well defined and has a meromorphic continuation having no zero in the half plane $\Re z < \frac{1}{2}$ for any $x \in]0, 1]$.

Therefore we would have from the HLR invariant conjecture 5.1.1 and the fundamental index conjecture 3.6

$$0 < x \leq 1 \Rightarrow \eta\left(f_x\right) = \alpha\left(f_x\right) \geq \frac{1}{2}$$

and I claim that letting $x \to 0$ the inequality still holds i.e.

$$\lim_{x \to 0} \eta\left(f_x\right) = \eta\left(\Phi\right) = \alpha\left(\Phi\right) \geq \frac{1}{2}$$

next from the Riemann functional equation we have $\eta\left(\Phi\right) \leq \frac{1}{2}$ hence we would have

$$\eta\left(\Phi\right) = \frac{1}{2}$$

meaning that RH is true.

References

[Cl1] B. Cloitre, Good variation theory: a Tauberian approach to RH, International Journal of Mathematics and Computer Science, Vol.2, 2016

[Cl2] B. Cloitre, Good variation theory, personal web page (2016)